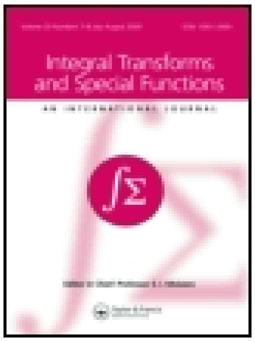

Integral Transforms and Special Functions

ISSN: 1065-2469 (Print) 1476-8291 (Online) Journal homepage: https://www.tandfonline.com/loi/gitr20

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INTEGRAL TRANSFORMS AND SPECIAL FUNCTIONS
https://doi.org/10.1080/10652469.2020.1737529

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Research Article

Macdonald's identities and integral representations of products of Airy functions

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ABSTRACT

In this paper, using the Macdonald's identities for the products of modified Bessel functions of first and second kinds, we derive new integral representations for the products of Airy functions and their derivatives. Manipulating the integrands of Macdonald's identities with various integral representations lead us to get new representations for the products of Airy functions and their derivatives.

ARTICLE HISTORY

Received 1 December 2019
Accepted 28 February 2020

KEYWORDS

Bessel functions; Macdonald's identities; Laplace transform; Airy function

AMS CLASSIFICATIONS

33C10; 44A10

1. Introduction

In view of the modified Bessel functions of first kind I_ν

$$I_\nu(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)}, \tag{1.1}$$

and the modified Bessel functions of second kind K_ν (Macdonald function)

$$K_\nu(\xi) = \frac{\pi}{2}(\pi\nu)[I_{-\nu}(\xi) - I_\nu(\xi)], \quad \nu \notin \mathbb{Z}, \tag{1.2}$$

Hector Munro Macdonald, in the late nineteenth century, introduced the following identities (Macdonald's identities) for the products of modified Bessel functions of first and second kinds. The first and second identities are considered as [1,p .712, 6.653(1,2), 2, p. 53 (36,37)]

$$I_\nu\Big(\min(w,z)\Big) K_\nu\Big(\max(w,z)\Big) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left[t + \frac{z^2+w^2}{t}\right]} I_\nu\left(\frac{zw}{t}\right)\frac{\mathrm{d}t}{t},$$
$$|\Re(\nu)| < 1, \quad z, w > 0, \tag{1.3}$$

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$$K_\nu(z)K_\nu(w) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left[t+\frac{z^2+w^2}{t}\right]} K_\nu\left(\frac{zw}{t}\right)\frac{\mathrm{d}t}{t},$$
$$|\arg(z)| < \pi,\ |\arg(w)| < \pi,\ |\arg(z+w)| < \frac{\pi}{4}. \tag{1.4}$$

Since the Airy functions of first and second kinds along with their derivatives are presented in terms of the modified Bessel functions (see also [3–5] for alternative representations in terms of the other types of Bessel functions)

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}(z) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}(z) - I_{\frac{1}{3}}(z)\right] = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)\mathrm{d}t, \quad z = \frac{2}{3}x^{\frac{3}{2}}, \tag{1.5}$$

$$\mathrm{Bi}(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}(z) + I_{\frac{1}{3}}(z)\right] = \frac{1}{\pi}\int_0^\infty \left(e^{xt-\frac{t^3}{3}} + \sin\left(xt + \frac{t^3}{3}\right)\right)\mathrm{d}t, \quad z = \frac{2}{3}x^{\frac{3}{2}}, \tag{1.6}$$

$$\mathrm{Ai}'(x) = -\frac{x}{\pi\sqrt{3}}K_{\frac{2}{3}}(z) = -\frac{x}{3}\left[I_{-\frac{2}{3}}(z) - I_{\frac{2}{3}}(z)\right], \quad z = \frac{2}{3}x^{\frac{3}{2}}, \tag{1.7}$$

$$\mathrm{Bi}'(x) = \frac{x}{\sqrt{3}}\left[I_{-\frac{2}{3}}(z) + I_{\frac{2}{3}}(z)\right], \quad z = \frac{2}{3}x^{\frac{3}{2}}, \tag{1.8}$$

the integral representations of products of Airy functions can be derived from the Bessel functions. The integral representations of products of Airy functions have been studied in the literature and some representations have been derived in terms of the elementary and special functions, see the trilogy of Reid [6–8], Varlamov's works [9–13] and other contributions [14–19]. It is of interest to find more integral representations for the products of Airy function over the literature, particularly in terms of the Bessel functions. In this sense, we employ the Macdonald's identities and replace the associated integrands with suitable integral representations (here the inverse Laplace transforms of desired functions) and derive new identities for the products of the modified Bessel functions. By using relations (1.5)–(1.8) and tables of integral transforms, the integral representations of products of Airy functions and their derivatives are consequently constructed.

2. Integral representations of products of Airy functions

2.1. Integral representations for products of Macdonald functions

Lemma 2.1: *The following integral representation holds for the products of modified Bessel functions of second kind*

$$K_\nu(z)K_\nu(w) = \int_0^\infty K_0\left(\sqrt{z^2 + w^2 + 2zw\cosh(\xi)}\right)\cosh(\nu\xi)\,\mathrm{d}\xi, \quad z, w > 0. \tag{2.1}$$

Proof: By replacing the following representation for the modified Bessel functions of second kind [1, p. 917, 8.432(1)]

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(\xi)}\cosh(\nu\xi)\,\mathrm{d}\xi, \quad z > 0, \tag{2.2}$$

and substituting into (1.4) we get

$$K_\nu(z)K_\nu(w) = \frac{1}{2}\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left[t+\frac{z^2+w^2+2zw\cosh(\xi)}{t}\right]} \cosh(\nu\xi)\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t}. \tag{2.3}$$

The result is obtained by employing the following representation for the modified Bessel functions of second kind

$$2K_0(2\sqrt{pq}) = \int_0^\infty e^{-pt-\frac{q}{t}}\frac{\mathrm{d}t}{t}, \quad p,q>0. \tag{2.4}$$

■

Theorem 2.2: *For $x \geq 0$, the following integral representation holds for the products of Airy functions*

$$\mathrm{Ai}^2(x) = \frac{1}{4\pi^2}\int_{4\sqrt{x}}^\infty K_0\left(\frac{t^3}{12}-xt\right)\frac{t^2-8x}{\sqrt{t^2-16x}}\,\mathrm{d}t. \tag{2.5}$$

Proof: Setting $z=w$ and $\nu=1/3$ in Lemma 2.1, and considering relation (1.5), we obtain

$$\frac{3\pi^2}{x}\mathrm{Ai}^2(x) = \int_0^\infty K_0\left(\frac{4}{3}x^{\frac{3}{2}}\cosh\left(\frac{\xi}{2}\right)\right)\cosh\left(\frac{\xi}{3}\right)\,\mathrm{d}\xi. \tag{2.6}$$

Changing of variables $\cosh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$ and applying a little algebra, we get the result. ■

Theorem 2.3: *For $x \geq 0$, the following integral representation holds for the products of derivatives of Airy functions*

$$\mathrm{Ai}'^2(x) = \frac{1}{16\pi^2}\int_{4\sqrt{x}}^\infty K_0\left(\frac{t^3}{12}-xt\right)\frac{t^4+32x^2-16xt^2}{\sqrt{t^2-16x}}\,\mathrm{d}t. \tag{2.7}$$

Proof: By setting $z=w$ and $\nu=2/3$ in Lemma 2.1, and using relation (1.7), we have

$$\frac{3\pi^2}{x^2}\mathrm{Ai}'^2(x) = \int_0^\infty K_0\left(\frac{4}{3}x^{\frac{3}{2}}\cosh\left(\frac{\xi}{2}\right)\right)\cosh\left(\frac{2\xi}{3}\right)\,\mathrm{d}\xi. \tag{2.8}$$

Changing of variables $\cosh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$ and applying a little algebra, we obtain (2.7). ■

Lemma 2.4: *For $|\Re(\nu)|<1$, the following integral representation holds for the products of modified Bessel functions of second kind*

$$K_\nu(z)K_\nu(w) = \frac{\pi}{4\sin(\nu\pi)}\int_{\frac{z^2+w^2}{2zw}}^\infty \frac{1}{\sqrt{u^2-1}}\left[(u+\sqrt{u^2-1})^\nu - (u+\sqrt{u^2-1})^{-\nu}\right] \times J_0(\sqrt{2zwu-z^2-w^2})\,\mathrm{d}u, \quad z,w>0. \tag{2.9}$$

Proof: First, we recall the Bessel function of first kind

$$J_\nu(\xi)=\sum_{k=0}^{\infty}\frac{(-1)^k\xi^{2k+\nu}}{2^{2k+\nu}k!\Gamma(\nu+k+1)}, \tag{2.10}$$

and replace a part of integrand (1.4), i.e. the function $\frac{1}{t}e^{-\frac{z^2+w^2}{2t}}$, with the following integral representation as the Laplace transform of J_0

$$\frac{1}{t}e^{-\frac{z^2+w^2}{2t}}=\int_0^\infty e^{-\xi t}J_0(\sqrt{2(z^2+w^2)\xi})\,\mathrm{d}\xi, \tag{2.11}$$

to obtain

$$K_\nu(z)K_\nu(w)=\frac{1}{2}\int_0^\infty\int_0^\infty e^{-t[\frac{1}{2}+\xi]}K_\nu\left(\frac{zw}{t}\right)J_0\left(\sqrt{2(z^2+w^2)\xi}\right)\,\mathrm{d}\xi\,\frac{\mathrm{d}t}{t}. \tag{2.12}$$

Next, by considering the following relation for the Laplace transform of modified Bessel functions of second kind subject to $\Re(a+u)>0$, [20, p. 349, 3.16.1(1)]

$$\begin{aligned}\mathcal{L}\{K_\nu(at);u\}&=\int_0^\infty e^{-ut}K_\nu(at)\,\mathrm{d}t\\&=\frac{\pi}{2\sin(\nu\pi)}\frac{1}{\sqrt{u^2-a^2}}\left[a^{-\nu}(u+\sqrt{u^2-a^2})^\nu-a^\nu(u+\sqrt{u^2-a^2})^{-\nu}\right],\end{aligned} \tag{2.13}$$

and incorporating the following identity as the Laplace transform of function $f\left(\frac{1}{t}\right)$, simultaneously, [21, p. 76, Table A(A-8)]

$$\mathcal{L}\left\{f\left(\frac{1}{t}\right);s\right\}=\frac{1}{\sqrt{s}}\int_0^\infty\sqrt{u}J_1(2\sqrt{su})\mathcal{L}\{f(t);u\}\,\mathrm{d}u, \tag{2.14}$$

for $a=zw$ and $s=\xi+\frac{1}{2}$, we get

$$\begin{aligned}K_\nu(z)K_\nu(w)=&\frac{\pi}{4\sin(\nu\pi)}\int_0^\infty\int_0^\infty J_0\left(\sqrt{2(z^2+w^2)\xi}\right)\sqrt{\frac{u}{\frac{1}{2}+\xi}}J_1\left(2\sqrt{u\left(\xi+\frac{1}{2}\right)}\right)\\&\times\frac{1}{\sqrt{u^2-z^2w^2}}\left[(zw)^{-\nu}(u+\sqrt{u^2-z^2w^2})^\nu\right.\\&\left.-(zw)^\nu(u+\sqrt{u^2-z^2w^2})^{-\nu}\right]\,\mathrm{d}\xi\,\mathrm{d}u.\end{aligned} \tag{2.15}$$

Here, we use the following representation [1, p. 693, 6.596(6)](H is the Heaviside function and $\Re(\mu)>\Re(\nu)>-1$)

$$\begin{aligned}&\int_0^\infty J_\nu(\beta x)\frac{J_\mu(\alpha\sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^\mu}}x^{\nu+1}\,\mathrm{d}x\\&=\frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2-\beta^2}}{z}\right)^{\mu-\nu-1}J_{\mu-\nu-1}(z\sqrt{\alpha^2-\beta^2})H(\alpha-\beta),\end{aligned} \tag{2.16}$$

to rewrite the above equation as

$$K_\nu(z)K_\nu(w) = \frac{\pi}{4\sin(\nu\pi)} \int_{\frac{z^2+w^2}{2}}^{\infty} \frac{1}{\sqrt{u^2 - z^2w^2}} \Big[(zw)^{-\nu}(u + \sqrt{u^2 - z^2w^2})^{\nu} - (zw)^{\nu}(u + \sqrt{u^2 - z^2w^2})^{-\nu} \Big] J_0(\sqrt{2u - (z^2 + w^2)})\, \mathrm{d}u, \tag{2.17}$$

or equivalently

$$K_\nu(z)K_\nu(w) = \frac{\pi}{4\sin(\nu\pi)} \int_{\frac{z^2+w^2}{2zw}}^{\infty} \frac{1}{\sqrt{u^2 - 1}} \Big[(u + \sqrt{u^2 - 1})^{\nu} - (u + \sqrt{u^2 - 1})^{-\nu} \Big] \times J_0(\sqrt{2zwu - z^2 - w^2})\, \mathrm{d}u. \tag{2.18}$$

■

Lemma 2.5 ([22, p. 140, Prob. 7(iii)]): *For* $|\Re(\nu)| < 1$, *the following integral representation holds for the products of modified Bessel functions of second kind*

$$K_\nu(z)K_\nu(w) = \frac{\pi}{2\sin(\nu\pi)} \int_{\ln(\frac{w}{z})}^{\infty} J_0\left(\sqrt{2zw\cosh(t) - z^2 - w^2}\right) \sinh(\nu t)\, \mathrm{d}t, \quad z, w > 0. \tag{2.19}$$

Proof: Setting $u = \cosh(t)$ in Lemma 2.4, we get the result. ■

Theorem 2.6: *For* $x \geq 0$, *the following integral representation holds for the products of Airy functions*

$$\mathrm{Ai}^2(x) = \frac{1}{4\pi\sqrt{3}} \int_0^{\infty} t J_0\left(xt + \frac{t^3}{12}\right) \mathrm{d}t. \tag{2.20}$$

Proof: Setting $z = w$ and $\nu = 1/3$ in Lemma 2.5, and considering relation (1.5), we obtain

$$\frac{3\pi^2}{x}\mathrm{Ai}^2(x) = \frac{\pi}{\sqrt{3}} \int_0^{\infty} J_0\left(\frac{4}{3}x^{\frac{3}{2}} \sinh\left(\frac{\xi}{2}\right)\right) \sinh\left(\frac{\xi}{3}\right) \mathrm{d}\xi. \tag{2.21}$$

The result is obtained after changing of variables $\sinh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$. ■

Theorem 2.7: *For* $x \geq 0$, *the following integral representation holds for the products of derivatives of Airy functions*

$$\mathrm{Ai}'^2(x) = \frac{1}{16\pi\sqrt{3}} \int_0^{\infty} t(t^2 + 8x) J_0\left(xt + \frac{t^3}{12}\right) \mathrm{d}t. \tag{2.22}$$

Proof: We first set $z = w$ and $\nu = 2/3$ in Lemma 2.5, then by using relation (1.7) we obtain

$$\frac{3\pi^2}{x^2}\mathrm{Ai}'^2(x) = \frac{\pi}{\sqrt{3}} \int_0^{\infty} J_0\left(\frac{4}{3}x^{\frac{3}{2}} \sinh\left(\frac{\xi}{2}\right)\right) \sinh\left(\frac{2\xi}{3}\right) \mathrm{d}\xi. \tag{2.23}$$

The result is derived after changing of variables $\sinh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$. ■

Lemma 2.8: *For $|\Re(\nu)| < 1$, the following integral representation holds for the products of modified Bessel functions of second kind*

$$K_\nu(z)K_\nu(w) = \frac{\pi}{2}\sin(\nu\pi)I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big) - \frac{\pi}{2}\int_0^\infty \frac{\xi}{\xi^2+1}\Big(\sin(\nu\pi)[Y_\nu(z\xi)Y_\nu(w\xi)] + \cos(\nu\pi)[J_\nu(w\xi)Y_\nu(z\xi)+J_\nu(z\xi)Y_\nu(w\xi)]\Big)\,\mathrm{d}\xi, \tag{2.24}$$

where Y_ν is the Bessel functions of second kind (Neumann function) given by

$$Y_\nu(\xi) = (\pi\nu)[J_\nu(\xi)\cos(\pi\nu) - J_{-\nu}(\xi)], \quad \nu \notin \mathbb{Z}. \tag{2.25}$$

Proof: By replacing the function $\frac{1}{t}e^{-\frac{z^2+w^2}{2t}}K_\nu\left(\frac{zw}{t}\right)$ with the following representation [23, p. 237, 3.16.3(27)]

$$\frac{1}{t}e^{-\frac{z^2+w^2}{2t}}K_\nu\left(\frac{zw}{t}\right) = -\frac{\pi}{2}\int_0^\infty e^{-\xi t}\Big(\sin(\nu\pi)[J_\nu(\sqrt{2}w\sqrt{\xi})J_\nu(\sqrt{2}z\sqrt{\xi}) - Y_\nu(\sqrt{2}z\sqrt{\xi})Y_\nu(\sqrt{2}w\sqrt{\xi})] + \cos(\nu\pi)[J_\nu(\sqrt{2}w\sqrt{\xi})Y_\nu(\sqrt{2}z\sqrt{\xi}) + J_\nu(\sqrt{2}z\sqrt{\xi})Y_\nu(\sqrt{2}w\sqrt{\xi})]\Big)\,\mathrm{d}\xi, \tag{2.26}$$

and substituting into (1.4) we gain

$$K_\nu(z)K_\nu(w) = -\frac{\pi}{4}\int_0^\infty\int_0^\infty e^{-(\xi+\frac{1}{2})t} \times\Big(\sin(\nu\pi)[J_\nu(\sqrt{2}w\sqrt{\xi})J_\nu(\sqrt{2}z\sqrt{\xi}) - Y_\nu(\sqrt{2}w\sqrt{\xi})Y_\nu(\sqrt{2}z\sqrt{\xi})] + \cos(\nu\pi)[J_\nu(\sqrt{2}w\sqrt{\xi})Y_\nu(\sqrt{2}z\sqrt{\xi}) + J_\nu(\sqrt{2}z\sqrt{\xi})Y_\nu(\sqrt{2}w\sqrt{\xi})]\Big)\,\mathrm{d}t\,\mathrm{d}\xi, \tag{2.27}$$

or equivalently

$$K_\nu(z)K_\nu(w) = \frac{\pi}{2}\int_0^\infty \frac{\xi}{\xi^2+1}\Big(\sin(\nu\pi)[J_\nu(w\xi)J_\nu(z\xi) - Y_\nu(w\xi)Y_\nu(z\xi)] + \cos(\nu\pi)[J_\nu(w\xi)Y_\nu(z\xi)+J_\nu(z\xi)Y_\nu(w\xi)]\Big)\,\mathrm{d}\xi. \tag{2.28}$$

At this stage, by considering the following identity for the products of I_ν and K_ν [24, p. 82, 10(12)]

$$\int_0^\infty \frac{\xi}{\xi^2+1}J_\nu(w\xi)J_\nu(z\xi)d\xi = I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big), \tag{2.29}$$

we rewrite the above relation as

$$
\begin{aligned}
K_\nu(z)K_\nu(w) = {} & \frac{\pi}{2}\sin(\nu\pi)I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big) \\
& - \frac{\pi}{2}\int_0^\infty \frac{\xi}{\xi^2+1}\Big(\sin(\nu\pi)[Y_\nu(w\xi)Y_\nu(z\xi)] \\
& + \cos(\nu\pi)[J_\nu(w\xi)Y_\nu(z\xi)+J_\nu(z\xi)Y_\nu(w\xi)]\Big)d\xi, \qquad (2.30)
\end{aligned}
$$

which completes the proof. ■

For showing more results, we first substitute $z = w$ and $\nu = \pm 1/3, \nu = \pm 2/3$ into Lemma 2.8, and then we add and subtract the associated relations to get the following theorems, respectively.

Theorem 2.9: *For $x > 0$, the following integral representation holds for the products of Airy functions*

$$
\begin{aligned}
\begin{bmatrix} \mathrm{Ai}^2(x) \\ \mathrm{Ai}(x)\mathrm{Bi}(x) \end{bmatrix} = {} & \begin{bmatrix} \dfrac{x}{11\sqrt{3}\pi} \\ \dfrac{x}{3\pi} \end{bmatrix} \int_0^\infty \frac{\xi}{\xi^2+1}\left[Y^2_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) \mp Y^2_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right] \mathrm{d}\xi \\
& - \begin{bmatrix} \dfrac{2x}{33\pi} \\ \dfrac{2x}{3\sqrt{3}\pi} \end{bmatrix} \int_0^\infty \frac{\xi}{\xi^2+1}\left[J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) Y_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right. \\
& \left. \pm J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) Y_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right] \mathrm{d}\xi. \qquad (2.31)
\end{aligned}
$$

Theorem 2.10: *For $x > 0$, the following integral representation holds for the products of derivatives of Airy functions*

$$
\begin{aligned}
\begin{bmatrix} \mathrm{Ai}'^2(x) \\ \mathrm{Ai}'(x)\mathrm{Bi}'(x) \end{bmatrix} = {} & \begin{bmatrix} \dfrac{x^2}{11\sqrt{3}\pi} \\ -\dfrac{x^2}{3\pi} \end{bmatrix} \int_0^\infty \frac{\xi}{\xi^2+1}\left[Y^2_{-\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) \mp Y^2_{\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right] \mathrm{d}\xi \\
& + \begin{bmatrix} \dfrac{2x^2}{33\pi} \\ -\dfrac{2x^2}{3\sqrt{3}\pi} \end{bmatrix} \int_0^\infty \frac{\xi}{\xi^2+1}\left[J_{-\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) Y_{-\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right. \\
& \left. \pm J_{\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right) Y_{\frac{2}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\xi\right)\right] \mathrm{d}\xi. \qquad (2.32)
\end{aligned}
$$

Lemma 2.11: *For $|\Re(\nu)| < 1$, the following integral representation holds for the products of modified Bessel functions of second kind*

$$\begin{aligned} K_\nu(z)K_\nu(w) &= \frac{\pi}{4\sin(\nu\pi)zw}\int_{\frac{z^2+w^2}{2zw}}^{\infty}\sqrt{2zwu-z^2-w^2}J_1(\sqrt{2zwu-z^2-w^2}) \\ &\times \frac{u}{\sqrt{(u^2-1)^3}}\left[\left(u+\sqrt{u^2-1}\right)^{\nu}-\left(u+\sqrt{u^2-1}\right)^{-\nu}\right]\mathrm{d}u \\ &- \frac{\nu\pi}{4\sin(\nu\pi)}\int_{\frac{z^2+w^2}{2zw}}^{\infty}\sqrt{2zwu-z^2-w^2}J_1(\sqrt{2zwu-z^2-w^2}) \\ &\times \frac{1}{\sqrt{u^2-1}}\left(1+\frac{u}{\sqrt{u^2-1}}\right) \\ &\times \left[\left(u+\sqrt{u^2-1}\right)^{\nu-1}+\left(u+\sqrt{u^2-1}\right)^{-\nu-1}\right]\mathrm{d}u. \end{aligned} \tag{2.33}$$

Proof: Using relation (2.14) as the Laplace transform of function $f\left(\frac{1}{t}\right)$ and considering the function $f(t) = te^{-\frac{z^2+w^2}{2}t}K_\nu(zwt)$, we can present the second Macdonald's identity as

$$\begin{aligned} K_\nu(z)K_\nu(w) &= \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left[t+\frac{z^2+w^2}{t}\right]}K_\nu\left(\frac{zw}{t}\right)\frac{\mathrm{d}t}{t}, \\ &= \frac{1}{2}\mathcal{L}\left\{f\left(\frac{1}{t}\right);\frac{1}{2}\right\} = \frac{1}{\sqrt{2}}\int_0^\infty \sqrt{u}J_1(\sqrt{2u})\mathcal{L}\{f(t);u\}\,\mathrm{d}u, \end{aligned} \tag{2.34}$$

where [23, p. 349, 3.16.1(1)]

$$\begin{aligned} \mathcal{L}\{f(t);u\} &= \frac{\pi}{2\sin(\nu\pi)}\frac{u+\frac{z^2+w^2}{2}}{\sqrt{\left(\left(u+\frac{z^2+w^2}{2}\right)^2-a^2\right)^3}} \\ &\times \left[a^{-\nu}\left(u+\frac{z^2+w^2}{2}+\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}\right)^{\nu}\right. \\ &\left. - a^{\nu}\left(\left(u+\frac{z^2+w^2}{2}\right)+\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}\right)^{-\nu}\right] \\ &- \frac{\pi}{2\sin(\nu\pi)}\frac{1}{\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}} \\ &\times \left[\nu a^{-\nu}\left(u+\frac{z^2+w^2}{2}+\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}\right)^{\nu-1}\right. \end{aligned}$$

$$\times\left(1+\frac{u+\frac{z^2+w^2}{2}}{\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}}\right)$$
$$+\nu a^{\nu}\left(\left(u+\frac{z^2+w^2}{2}\right)+\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}\right)^{-\nu-1}$$
$$\times\left(1+\frac{u+\frac{z^2+w^2}{2}}{\sqrt{\left(u+\frac{z^2+w^2}{2}\right)^2-a^2}}\right)\Bigg],\quad a=zw. \tag{2.35}$$

At this point, we use the suitable change of variables to obtain the result. ■

Lemma 2.12: *For $|\Re(\nu)|<1$, the following integral representation holds for the products of modified Bessel functions of second kind*

$$K_\nu(z)K_\nu(w)=\frac{\pi}{2\sin(\nu\pi)}\int_{\ln\left(\frac{w}{z}\right)}^{\infty}\sqrt{2zw\cosh(t)-z^2-w^2}J_1(\sqrt{2zw\cosh(t)-z^2-w^2})$$
$$\times\left[\frac{\cosh(t)}{zw\sinh^2(t)}\sinh(\nu t)-\nu e^{-t}(1+\coth(t))\cosh(\nu t)\right]\mathrm{d}t. \tag{2.36}$$

Proof: Setting $u=\cosh(t)$ in Lemma 2.11, we get the result. ■

Theorem 2.13: *For $x\geq 0$, the following integral representation holds for the products of Airy functions*

$$\mathrm{Ai}^2(x)=\frac{12}{\sqrt{3}\pi}\int_0^\infty tJ_1\left(xt+\frac{t^3}{12}\right)\frac{(t^2+16x)(t^2+4x)^2+(t^3+12xt)^2}{(t^2+16x)(t^2+4x)^2(t^3+12xt)}\mathrm{d}t$$
$$-\frac{1}{12\sqrt{3}\pi}\int_0^\infty e^{-t}\frac{t^2+8x}{\sqrt{t^2+16x}}\left(xt+\frac{t^3}{12}\right)J_1\left(xt+\frac{t^3}{12}\right)$$
$$\times\left(1+\frac{(t^2+16x)(t^2+4x)^2+(t^3+12xt)^2}{\sqrt{(t^2+16x)}(t^2+4x)(t^3+12xt)}\right)\mathrm{d}t. \tag{2.37}$$

Proof: Setting $z=w$ and $\nu=1/3$ in Lemma 2.12, and considering relation (1.5), we obtain

$$\frac{3\pi^2}{x}\mathrm{Ai}^2(x)=\frac{\pi}{\sqrt{3}}\int_0^\infty\sqrt{\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)}J_1\left(\sqrt{\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)}\right)$$
$$\times\left[\frac{9\cosh(\xi)}{4x^3\sinh^2(\xi)}\sinh\left(\frac{\xi}{3}\right)-\frac{1}{3}e^{-t}(1+\coth(\xi))\cosh\left(\frac{\xi}{3}\right)\right]\mathrm{d}\xi. \tag{2.38}$$

By the same procedure to the previous proofs, we can get the result by applying the change of variables $\sinh\left(\frac{\xi}{6}\right)=\frac{t}{4\sqrt{x}}$. ■

Theorem 2.14: *For $x \geq 0$, the following integral representation holds for the products of derivatives of Airy functions*

$$\mathrm{Ai}'^2(x) = \frac{3}{\sqrt{3}\pi}\int_0^\infty t(t^2+8x)J_1\left(xt+\frac{t^3}{12}\right)\frac{(t^2+16x)(t^2+4x)^2+(t^3+12xt)^2}{(t^2+16x)(t^2+4x)^2(t^3+12xt)}\,\mathrm{d}t$$
$$-\frac{1}{24\sqrt{3}\pi}\int_0^\infty e^{-t}\frac{t^4+32x^2+16xt^2}{\sqrt{t^2+16x}}\left(xt+\frac{t^3}{12}\right)J_1\left(xt+\frac{t^3}{12}\right)$$
$$\times\left(1+\frac{(t^2+16x)(t^2+4x)^2+(t^3+12xt)^2}{\sqrt{(t^2+16x)}(t^2+4x)(t^3+12xt)}\right)\mathrm{d}t. \tag{2.39}$$

Proof: We set $z = w$ and $\nu = 2/3$ in Lemma 2.12, and use relation (1.7). So, we have

$$\frac{3\pi^2}{x^2}\mathrm{Ai}'^2(x) = \frac{\pi}{\sqrt{3}}\int_0^\infty \sqrt{\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)}J_1\left(\sqrt{\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)}\right)$$
$$\times\left[\frac{9\cosh(\xi)}{4x^3\sinh^2(\xi)}\sinh\left(\frac{2\xi}{3}\right)-\frac{2}{3}e^{-t}(1+\coth(\xi))\cosh\left(\frac{2\xi}{3}\right)\right]\mathrm{d}\xi. \tag{2.40}$$

The result is obtained by changing of variables $\sinh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$. ■

2.2. Integral representations for products of I_ν and K_ν

Lemma 2.15: *For $|\Re(\nu)| < 1$, the following integral representation holds for the products of modified Bessel functions*

$$I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big)$$
$$=\frac{1}{2}\int_{\frac{z^2+w^2}{2zw}}^\infty \frac{1}{\sqrt{u^2-1}}(u+\sqrt{u^2-1})^{-\nu}J_0(\sqrt{2zwu-z^2-w^2})\,\mathrm{d}u, \quad z,w>0. \tag{2.41}$$

Proof: By the same proof to that of Lemma 2.4 and considering the following relation for the Laplace transform of modified Bessel function of first kind [20, p. 313, 3.15.1(1)]

$$\mathcal{L}\{I_\nu(at);u\} = \int_0^\infty e^{-ut}I_\nu(at)\,\mathrm{d}t = \frac{a^\nu}{\sqrt{u^2-a^2}}(u+\sqrt{u^2-a^2})^{-\nu}, \tag{2.42}$$

we get the result. ■

Lemma 2.16: *For $|\Re(\nu)| < 1$, the following integral representation holds for the products of modified Bessel functions*

$$I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big)$$
$$=\frac{1}{2}\int_{\ln\left[\frac{\max(w,z)}{\min(w,z)}\right]}^\infty J_0(\sqrt{2zw\cosh(t)-z^2-w^2})e^{-\nu t}\,\mathrm{d}t, \quad z,w>0. \tag{2.43}$$

Proof: Setting $u = \cosh(t)$ in Lemma 2.15, we get the result. ■

Theorem 2.17: *For* $x \geq 0$, *the following integral representation holds for the products of Airy functions*

$$\mathrm{Ai}(x)\mathrm{Bi}(x) = \frac{1}{4\pi}\int_0^\infty J_0\left(xt + \frac{t^3}{12}\right)\frac{t^2+8x}{\sqrt{t^2+16x}}\,\mathrm{d}t. \tag{2.44}$$

Proof: We set $z = w$ and $\nu = \pm 1/3$ in Lemma 2.16 and add the associated relations. In this sense, we consider the relation (1.6) and obtain

$$\frac{3\pi}{x}\mathrm{Ai}(x)\mathrm{Bi}(x) = \int_0^\infty J_0\left(\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)\right)\cosh\left(\frac{\xi}{3}\right)\,\mathrm{d}\xi, \tag{2.45}$$

which implies relation (2.44). ■

Theorem 2.18: *For* $x \geq 0$, *the following integral representation holds for the products of derivatives of Airy functions*

$$\mathrm{Ai}'(x)\mathrm{Bi}'(x) = -\frac{1}{16\pi}\int_0^\infty J_0\left(xt + \frac{t^3}{12}\right)\frac{t^4+32x^2+16xt^2}{\sqrt{t^2+16x}}\,\mathrm{d}t. \tag{2.46}$$

Proof: By the same procedure to previous theorem, we set $z = w$ and $\nu = \pm 2/3$ in Lemma 2.16, and then we add the associated relations. We use relations (1.7) and (1.8) to get

$$-\frac{3\pi}{x^2}\mathrm{Ai}'(x)\mathrm{Bi}'(x) = \int_0^\infty J_0\left(\frac{4}{3}x^{\frac{3}{2}}\sinh\left(\frac{\xi}{2}\right)\right)\cosh\left(\frac{2\xi}{3}\right)\,\mathrm{d}\xi. \tag{2.47}$$

Changing of variables $\sinh\left(\frac{\xi}{6}\right) = \frac{t}{4\sqrt{x}}$ and applying a little algebra, we obtain (2.46). ■

Lemma 2.19: *For* $|\Re(\nu)| < 1$, *the following integral representation holds for the products of modified Bessel functions*

$$I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big) = \int_0^\infty \sqrt{\frac{z^2+w^2}{2\xi+1}}K_1\Big(\sqrt{(2\xi+1)(z^2+w^2)}\Big) \times \Big[\mathrm{ber}_\nu^2(\sqrt{2zw\xi}) + \mathrm{bei}_\nu^2(\sqrt{2zw\xi})\Big]\,\mathrm{d}\xi, \quad z,w>0, \tag{2.48}$$

where the Thomson functions $\mathrm{ber}_\nu(z)$ *and* $\mathrm{bei}_\nu(z)$ *are shown by* [1, p. 944, 8.561(1,2)]

$$\mathrm{ber}_\nu(z) = \frac{1}{2}[J_\nu(ze^{3\pi i/4}) + J_\nu(ze^{-3\pi i/4})], \tag{2.49}$$

$$\mathrm{bei}_\nu(z) = \frac{1}{2i}[J_\nu(ze^{3\pi i/4}) - J_\nu(ze^{-3\pi i/4})]. \tag{2.50}$$

Proof: By considering the inverse Laplace transform of the function $\frac{1}{t}I_\nu\left(\frac{zw}{t}\right)$ in terms of the Thomson functions [23, p. 215, 3.15.1(4)]

$$\frac{1}{t}I_\nu\left(\frac{zw}{t}\right) = \int_0^\infty e^{-\xi t}\Big[\mathrm{ber}_\nu^2(\sqrt{2zw\xi}) + \mathrm{bei}_\nu^2(\sqrt{2zw\xi})\Big]\,\mathrm{d}\xi, \tag{2.51}$$

the first Macdonald's identity is changed to

$$I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big) = \frac{1}{2}\int_0^\infty\int_0^\infty e^{-t\left(\xi+\frac{1}{2}\right)}e^{-\frac{z^2+w^2}{2t}} \times\Big[\mathrm{ber}_\nu^2(\sqrt{2zw\xi}) + \mathrm{bei}_\nu^2(\sqrt{2zw\xi})\Big]\,\mathrm{d}t\,\mathrm{d}\xi, \quad z,w>0. \tag{2.52}$$

Now, by taking into account the following integral representation [1, p. 337, 3.324(1)]

$$\int_0^\infty e^{-\frac{\beta}{4t}-\gamma t}\,\mathrm{d}t = \sqrt{\frac{\beta}{\gamma}}K_1(\sqrt{\beta\gamma}), \quad \Re(\beta)\geq 0, \Re(\gamma)>0, \tag{2.53}$$

integral (2.52) is written as

$$I_\nu\Big(\min(w,z)\Big)K_\nu\Big(\max(w,z)\Big) = \int_0^\infty \sqrt{\frac{z^2+w^2}{2\xi+1}}K_1\Big(\sqrt{(2\xi+1)(z^2+w^2)}\Big) \times\Big[\mathrm{ber}_\nu^2(\sqrt{2zw\xi}) + \mathrm{bei}_\nu^2(\sqrt{2zw\xi})\Big]\,\mathrm{d}\xi, \quad z,w>0. \tag{2.54}$$

■

For showing more results, we set $z = w$ and $\nu = \pm 1/3$, $\nu = \pm 2/3$ in Lemma 2.19, and then we add and subtract the associated relations to state the following theorems.

Theorem 2.20: *For $x \geq 0$, the following integral representation holds for the products of Airy functions*

$$\mathrm{Ai}(x)\mathrm{Bi}(x) = \frac{2\sqrt{2}x^{5/2}}{9\pi}\int_0^\infty \frac{K_1\Big(\sqrt{8/9x^3(2\xi+1)}\Big)}{\sqrt{2\xi+1}} \times\Big[\mathrm{ber}_{-\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) + \mathrm{ber}_{\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) + \mathrm{bei}_{-\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) + \mathrm{bei}_{\frac{1}{3}}^2(\sqrt{8/9x^3\xi})\Big]\,d\xi, \tag{2.55}$$

$$\mathrm{Ai}^2(x) = \frac{2\sqrt{2}x^{5/2}}{9\sqrt{3}\pi}\int_0^\infty \frac{K_1\Big(\sqrt{8/9x^3(2\xi+1)}\Big)}{\sqrt{2\xi+1}} \times\Big[\mathrm{ber}_{-\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) - \mathrm{ber}_{\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) + \mathrm{bei}_{-\frac{1}{3}}^2(\sqrt{8/9x^3\xi}) - \mathrm{bei}_{\frac{1}{3}}^2(\sqrt{8/9x^3\xi})\Big]\,\mathrm{d}\xi. \tag{2.56}$$

Theorem 2.21: *For $x \geq 0$, the following integral representation holds for the products of derivatives of Airy functions*

$$\begin{aligned} \mathrm{Ai}'(x)\mathrm{Bi}'(x) = {} & -\frac{2\sqrt{2}x^{7/2}}{9\pi}\int_0^\infty \frac{K_1\left(\sqrt{8/9x^3(2\xi+1)}\right)}{\sqrt{2\xi+1}} \\ & \times \left[\mathrm{ber}^2_{-\frac{2}{3}}(\sqrt{8/9x^3\xi}) + \mathrm{ber}^2_{\frac{2}{3}}(\sqrt{8/9x^3\xi}) \right. \\ & \left. + \mathrm{bei}^2_{-\frac{2}{3}}(\sqrt{8/9x^3\xi}) + \mathrm{bei}^2_{\frac{2}{3}}(\sqrt{8/9x^3\xi})\right] \mathrm{d}\xi, \end{aligned} \tag{2.57}$$

$$\begin{aligned} \mathrm{Ai}'^2(x) = {} & \frac{2\sqrt{2}x^{7/2}}{9\sqrt{3}\pi}\int_0^\infty \frac{K_1\left(\sqrt{8/9x^3(2\xi+1)}\right)}{\sqrt{2\xi+1}} \\ & \times \left[\mathrm{ber}^2_{-\frac{2}{3}}(\sqrt{8/9x^3\xi}) - \mathrm{ber}^2_{\frac{2}{3}}(\sqrt{8/9x^3\xi}) \right. \\ & \left. + \mathrm{bei}^2_{-\frac{2}{3}}(\sqrt{8/9x^3\xi}) - \mathrm{bei}^2_{\frac{2}{3}}(\sqrt{8/9x^3\xi})\right] \mathrm{d}\xi. \end{aligned} \tag{2.58}$$

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The first author would like to acknowledge Shahrekord University for financial support under grant 97GRN1M1328.

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