OPTIMIZATION, 2016
VOL. 65, NO. 12, 2293–2308
http://dx.doi.org/10.1080/02331934.2016.1231187

Taylor & Francis
Taylor & Francis Group

A predictor-corrector infeasible-interior-point method for the Cartesian $P_*(\kappa)$-LCP over symmetric cones with $O\left(\sqrt{\mathrm{cond}(G)}(1+\kappa)^2 r \log \varepsilon^{-1}\right)$ iteration complexity

M. Sayadi Shahraki, H. Mansouri and M. Zangiabadi

Faculty of Mathematical Sciences, Department of Applied Mathematics, Shahrekord University, Shahrekord, Iran

ABSTRACT
In this paper, we present a predictor-corrector infeasible-interior-point method for symmetric cone linear complementarity problem (SCLCP) with the Cartesian $P_*(\kappa)$-property ($P_*(\kappa)$-SCLCP). This method is based on a wide neighbourhood, which is an even wider neighbourhood than the negative infinity neighbourhood. We show that the iteration-complexity bound of the proposed algorithm for a commutative class of search directions is $O\left(\sqrt{cond(G)}(1+\kappa)^2 r \log \varepsilon^{-1}\right)$, where $\mathrm{cond}(G)$ is the condition number of matrix G, κ is the handicap of the problem, r is the rank of the associated Euclidean Jordan algebra and $\varepsilon > 0$ is a given tolerance. To our knowledge, this is the best complexity result obtained so far for the wide neighbourhood infeasible-interior-point methods for the Cartesian $P_*(\kappa)$-SCLCPs.

ARTICLE HISTORY
Received 15 October 2015
Accepted 5 August 2016

KEYWORDS
Infeasible-interior-point method; wide neighborhood; linear complementarity problem; symmetric cone programming; $P_*(\kappa)$-property

1. Introduction

Interior-point methods (IPMs) that initiated by Karmarkar [1] play an important role in modern mathematical programming. They have been proposed for linear programming (LP), and then many of these methods are extended to symmetric cone programming (SCP).[2–8] SCP includes solving problems such as LP, semidefinite programming (SDP) and second-order cone programming. The foundation for solving these problems using IPMs was laid by Nesterov and Nemirovski [9]. The first extension of primal-dual IPMs for SCP was achieved by Nesterov and Todd [10,11].

Two popular neighbourhoods used in IPMs are so-called small neighbourhood and negative infinity wide neighbourhood. Ai [12] and Ai and Zhang [13] proposed a new class of wider neighbourhoods for LP and linear complementarity problems (LCPs), respectively, which is known as $\mathcal{N}(\tau, \beta)$ (see Section 3). Li and Terlaky [14] extended the Ai and Zhang's technique to SDP. In 2013, Liu et al. [15] extended the wide neighbourhood $\mathcal{N}(\tau, \beta)$ to SCP. Recently, Yang et al. [16] proposed a new approach in the complexity analysis of an infeasible-IPM for SCP based on the wide neighbourhood $\mathcal{N}(\tau, \beta)$ and improved the theoretical complexity bound in Liu et al. [15]. Motivated by these results, we present a predictor-corrector infeasible-interior-point algorithm for $P_*(\kappa)$-SCLCP. The current paper aims at modifying Yang et al.'s algorithm in [16] to gain a new class of second-order corrector interior point algorithm for $P_*(\kappa)$-SCLCP.

The class of $P_*(\kappa)$-matrices was first introduced by Kojima et al. [17]. Later, the Cartesian $P_*(\kappa)$-SCLCP introduced by Luo and Xiu [18]. The Cartesian $P_*(\kappa)$ class involves the Cartesian P class and turns out to be a special case in the Cartesian P_0 class. Several efficient algorithms have been

CONTACT H. Mansouri Mansouri@sci.sku.ac.ir

proposed for the Cartesian $P_*(\kappa)$-SCLCP and the Cartesian P-matrix SCLCP in [19–24]. Based on the Nesterov–Todd (NT) search direction, in [20,23,24] the authors proposed a class of polynomial interior point algorithms, which generates a sequence of iterates in the small neighbourhood of the central path. The first extension of infeasible-IPM based on the wide neighbourhood $\mathcal{N}(\tau, \beta)$ in [15] to the Cartesian $P_*(\kappa)$-SCLCP was achieved by Sayadi Shahraki et al. [22]. Furthermore, by using the NT search direction, the iteration complexity for this class of optimization problems is obtained as $O\left((1+\kappa)^3 r^2 \log \varepsilon^{-1}\right)$.[19,21] In [21], the iteration complexities for *xs* and *sx* search directions are obtained as $O\left((1+\kappa)^3 r^{2.5} \log \varepsilon^{-1}\right)$.

In this paper, we improve the iteration complexity for the NT search direction to $O\Big((1+\kappa)^2 r \log \varepsilon^{-1}\Big)$ and the iteration complexities for *xs* and *sx* search directions to $O\left((1+\kappa)^2 r^{3/2} \log \varepsilon^{-1}\right)$.

This paper is organized as follows: In Section 2, we give a brief introduction to Euclidean Jordan algebra and IPM for the Cartesian $P_*(\kappa)$-SCLCP. In Section 3, we present an interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP. In Section 4, we analyse the algorithm and obtain the currently best-known iteration bound for infeasible-IPMs for Cartesian $P_*(\kappa)$-SCLCP. Finally, some conclusions and remarks follow in section 5.

2. Preliminaries

2.1. Euclidean Jordan algebras and symmetric cones

In this section, we recall some concepts of Euclidean Jordan algebra and symmetric cones which are needed in this paper. For a comprehensive treatment of Euclidean Jordan algebras, the reader is referred to the monograph by Farut and Korány [25].

Definition 1: Let $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ be an n-dimensional inner product space over $\mathbb{R}$ and $\circ : (x, y) \mapsto x \circ y$ be a bilinear map from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{J}$. Then, the triple $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$ is a Euclidean Jordan algebra if it satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$, where $x^2 := x \circ x$;
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in \mathcal{J}$, where the inner product $\langle\cdot,\cdot\rangle$ is defined by $\langle x, y\rangle := \mathbf{Tr}(x \circ y)$ for any $x, y \in \mathcal{J}$.

Since '$\circ$' is bilinear for every $x \in \mathcal{J}$, there exists a linear operator $L(x)$ such that for every $y \in \mathcal{J}$, $L(x)y := x \circ y$. The vectors x and y are said to be operator commute if $L(x)L(y) = L(y)L(x)$. In other words, x and y are operator commute if $x \circ (y \circ z) = y \circ (x \circ z)$, for all $z \in \mathcal{J}$. Additionally, we define

$$Q(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$. $Q(x)$ is called the quadratic representation of x. In the following, we present some important properties of the quadratic representation.

Proposition 2.1 (Proposition III.2.2 in [25]): *Let $x, s \in int\,\mathcal{K}$. Then $Q(x)s \in int\,\mathcal{K}$.*

Lemma 2.2 (Lemma 28 in [4]): *Let $x, s \in int\,\mathcal{K}$ and p be invertible. Then $x \circ s = \mu e$ if and only if $Q(p)x \circ Q(p^{-1})s = \mu e$.*

Lemma 2.3 (Proposition 2.9 in [18]): *Let $x, s \in int\,\mathcal{K}$. If x and s are operator commute then $Q(x^{1/2})s = x \circ s$.*

For a Euclidean Jordan algebra $\mathcal{J}$, the corresponding cone of squares $\mathcal{K} := \{x^2 : x \in \mathcal{J}\}$ is a symmetric cone. A Jordan algebra has an identity element, if there exists a unique element e, such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. For any $x \in \mathcal{J}$, let k be the smallest integer such that the set

$\{e, x, \ldots, x^k\}$ is linearly dependent. Then, k is the degree of x which is denoted by $\deg(x)$. The rank of $\mathcal{J}$ is the largest $\deg(x)$ of any element $x \in \mathcal{J}$.

An element $c \in \mathcal{J}$ is said to be an idempotent if $c \neq 0$ and $c^2 = c$. An idempotent c is primitive if it is nonzero and cannot be expressed by sum of two other nonzero idempotents. Two idempotents c_i and c_l are said to be orthogonal if $c_i \circ c_l = 0$. We say that $\{c_1, c_2, \ldots, c_k\}$ is a Jordan frame if each c_i is a primitive idempotent, $c_i \circ c_l = 0$ for all $i \neq l$, and $\sum_{j=1}^{k} c_j = e$.

Theorem 2.4 (Theorem III.1.2 in [25]): *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r. For any $x \in \mathcal{J}$, there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \sum_{i=1}^{r} \lambda_i c_i. \tag{1}$$

Every λ_i is called an eigenvalue of x and (1) is the spectral decomposition of x. We denote $\lambda_{\min}$ $(\lambda_{\max})$ as the minimal (maximal) eigenvalue of x.

By using eigenvalues, we may extend the definition of any real-valued continuous function to elements of a Euclidean Jordan algebra. Particularly, we have some examples as follow:

Square root: $x^{1/2} := \sum_{i=1}^{r} \lambda_i^{1/2} c_i$ if all $\lambda_i \geq 0$;
Inverse: $x^{-1} := \sum_{i=1}^{r} \lambda_i^{-1} c_i$ if all $\lambda_i \neq 0$;
Trace: $\mathbf{Tr}(x) = \sum_{i=1}^{r} \lambda_i$;
Determinant: $\det(x) = \prod_{i=1}^{r} \lambda_i$;
Frobenius norm: $\|x\| := \sqrt{\langle x, x\rangle} = \left(\sum_{i=1}^{r} \lambda_i^2\right)^{1/2}$;
Metric projection: $x^+ = \sum_{i=1}^{r} \lambda_i^+ c_i$ where $\lambda_i^+ = \max\{\lambda_i, 0\}$ for $i = 1, 2, \ldots, r$. Moreover, $x^- = x - x^+$.

In the following, we recall two lemmas which are useful in the complexity analysis of the algorithm.

Lemma 2.5 (Lemma 2.15 in [7]): *Let $x \circ s \in int\, \mathcal{K}$, then $det(x) \neq 0$.*

Lemma 2.6 (Lemma 5.12 in [26]): *If $x, y \in \mathcal{J}$, then*

$$\left\|(x + y)^+\right\| \leq \left\|x^+ + y^+\right\| \leq \left\|x^+\right\| + \left\|y^+\right\|.$$

2.2. IPM for the Cartesian $P_*(\kappa)$-SCLCP

Let n_ν-dimensional space $\mathcal{J}_\nu$ be a Euclidean Jordan algebra and $\mathcal{K}_\nu$ is the corresponding symmetric cone with rank r_ν, for any $\nu \in \{1, 2, \ldots, N\}$. Let $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \cdots \times \mathcal{J}_N$ is the Cartesian product space with its cone of squares $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_N$ and the dimension and rank of $\mathcal{J}$ are $n = \sum_{\nu=1}^{N} n_\nu$ and $r = \sum_{\nu=1}^{N} r_\nu$, respectively. In this paper, we consider

SCLCP, given in the standard form

$$x \in \mathcal{K},\ s = \mathcal{A}(x) + q \in \mathcal{K},\ \langle x, s\rangle = 0, \tag{2}$$

where $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ is a given linear transformation and $q \in \mathcal{J}$.

We call SCLCP the Cartesian $P_*(\kappa)$-SCLCP if linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$ property for some nonnegative constant $\kappa \geq 0$, i.e. $\mathcal{A}(u) - v = 0$ implies

$$(1 + 4\kappa) \sum_{\nu \in I_+} \langle u^{(\nu)}, v^{(\nu)}\rangle + \sum_{\nu \in I_-} \langle u^{(\nu)}, v^{(\nu)}\rangle \geq 0,\ \forall\, u, v \in \mathcal{J},$$

where $I_+ = \{\nu\ :\ \langle u^{(\nu)}, v^{(\nu)}\rangle \geq 0\}$ and $I_- = \{\nu\ :\ \langle u^{(\nu)}, v^{(\nu)}\rangle < 0\}$.

Throughout the paper, we assume that the Cartesian $P_*(\kappa)$-SCLCP satisfies the interior-point condition (IPC), i.e. there exist $(x,\, s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K}$ such that $s = \mathcal{A}(x) + q$. Under the IPC, finding an optimal solution of (2) is equivalent to solving the following system:

$$\begin{aligned} \mathcal{A}(x) - s &= -q, \\ x \circ s &= 0, \\ x,\, s &\in \operatorname{int}\mathcal{K}, \end{aligned} \tag{3}$$

The basic idea of primal-dual IPMs is to replace the second equation in (3), the so-called complementary condition for the Cartesian $P_*(\kappa)$-SCLCP, by the parameterized equation $x \circ s = \tau\mu e$, with $\mu = \langle x,\, s\rangle/r$ which is called the duality gap and $\tau \in]0,\, 1[$ is called centring parameter. This yields the following system

$$\begin{aligned} \mathcal{A}(x) - s &= -q, \\ x \circ s &= \tau\mu e, \\ x,\, s &\in \operatorname{int}\mathcal{K}. \end{aligned} \tag{4}$$

A natural way to define a search direction is to follow Newton's approach and linearize the second equation in (4). This leads to the following system:

$$\begin{aligned} \mathcal{A}(\Delta x) - \Delta s &= \rho, \\ s \circ \Delta x + x \circ \Delta s &= \tau\mu e - x \circ s, \end{aligned} \tag{5}$$

where the residual is denoted by ρ and is defined as

$$\rho = s - \mathcal{A}(x) - q.$$

Due to the fact that x and s may be not operator commute in general, the system (5) does not always have a unique solution. Therefore, we restrict the scaling u belong to the set of all elements so that the scaled elements are operator commute, i.e.

$$\mathcal{C}(x,\, s) = \left\{u : u \in \operatorname{int}\mathcal{K} \text{ such that } Q(u)x \text{ and } Q(u^{-1})s \text{ are operator commute}\right\}.$$

In particular, for

$$u = \left[Q(x)^{\frac{1}{2}}\left(Q(x^{\frac{1}{2}})s\right)^{-\frac{1}{2}}\right]^{-1/2} = \left[Q(s^{-\frac{1}{2}})\left(Q(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}\right]^{-1/2},$$

we obtain the NT search direction. Moreover, for the choice of $u = s^{1/2}$ we get the *xs* search direction and for $u = x^{-1/2}$ we get the *sx* search direction.

Let $\widetilde{\mathcal{A}} = Q(u^{-1})\mathcal{A}Q(u^{-1})$, $\widetilde{x} = Q(u)x$, $\widetilde{s} = Q(u^{-1})s$, $\Delta\widetilde{x} = Q(u)\Delta x$, $\Delta\widetilde{s} = Q(u^{-1})\Delta s$ and $\widetilde{\rho} = Q(u^{-1})\rho$. With these notations and Lemma 2.2, the Newton system becomes

$$\begin{aligned} \widetilde{\mathcal{A}}(\Delta\widetilde{x}) - \Delta\widetilde{s} &= \widetilde{\rho}, \\ \widetilde{s} \circ \Delta\widetilde{x} + \widetilde{x} \circ \Delta\widetilde{s} &= \tau\mu e - \widetilde{x} \circ \widetilde{s}. \end{aligned} \tag{6}$$

3. Algorithm

Most IPMs are primal-dual path following methods, the iterates are confined to stay within a neighbourhood of the central path which is defined as

$$\mathcal{C} = \{(x,\, s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} :\ x \circ s = \mu e,\ \mu > 0\}.$$

The negative infinity neighbourhood which is a wide neighbourhood of the central path, is defined as follows:

$$\mathcal{N}_{\infty}^{-}(1-\gamma) = \{(x, s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} : \lambda_{\min}(w) \geq \gamma\mu\},$$

where $\gamma \in]0, 1[$ and $w = Q(x^{1/2})s$.

In this paper, we will restrict the iterates into the following wide neighbourhood of the central path for the Cartesian $P_*(\kappa)$-SCLCP, which introduced by Ai [12] for LP and Ai and Zhang [13] for LCP and Liu et al. [15] for SCP:

$$\mathcal{N}(\tau, \beta) = \left\{(x, s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} : \left\|(\tau\mu e - w)^{+}\right\| \leq \beta\tau\mu\right\}, \tag{7}$$

where $\beta, \tau \in]0, 1[$.

Remark 1: From the definition of $\mathcal{N}(\tau, \beta)$, it is obvious that

$$\mathcal{N}_{\infty}^{-}(1-\tau) \subseteq \mathcal{N}(\tau, \beta) \subseteq \mathcal{N}_{\infty}^{-}\left(1-\left(1-\beta\right)\tau\right).$$

To obtain the predictor directions, we need to solve the following two systems:

$$\begin{aligned} \widetilde{\mathcal{A}}(\Delta\widetilde{x}_1) - \Delta\widetilde{s}_1 &= \widetilde{\rho}, \\ \widetilde{s} \circ \Delta\widetilde{x}_1 + \widetilde{x} \circ \Delta\widetilde{s}_1 &= (\tau\mu e - \widetilde{x} \circ \widetilde{s})^{-} + \sqrt{r}\,(\tau\mu e - \widetilde{x} \circ \widetilde{s})^{+}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \widetilde{\mathcal{A}}(\Delta\widetilde{x}_2) - \Delta\widetilde{s}_2 &= 0, \\ \widetilde{s} \circ \Delta\widetilde{x}_2 + \widetilde{x} \circ \Delta\widetilde{s}_2 &= (\tau\mu e - \widetilde{x} \circ \widetilde{s})^{-} + \sqrt{r}\,(\tau\mu e - \widetilde{x} \circ \widetilde{s})^{+}. \end{aligned} \tag{9}$$

Then, we compute the maximum size $\delta \in [0, 1]$ that ensures

$$\mathbf{Tr}\left(\Delta\widetilde{x}_3 \circ \Delta\widetilde{s}_3\right) \geq -\tfrac{3}{5}(1+2\kappa)(1+\beta\tau)r\mu, \tag{10}$$

where

$$\begin{aligned} \Delta\widetilde{x}_3 &= \delta\Delta\widetilde{x}_1 + \left(1-\delta\right)\Delta\widetilde{x}_2, \\ \Delta\widetilde{s}_3 &= \delta\Delta\widetilde{s}_1 + \left(1-\delta\right)\Delta\widetilde{s}_2. \end{aligned} \tag{11}$$

The predictor directions are obtained in the same way as [16]. By using $\Delta\widetilde{x}_3$ and $\Delta\widetilde{s}_3$, we compute the corrector directions $\Delta\widetilde{x}^c$ and $\Delta\widetilde{s}^c$ as follows:

$$\begin{aligned} \widetilde{\mathcal{A}}(\Delta\widetilde{x}^c) - \Delta\widetilde{s}^c &= 0, \\ \widetilde{s} \circ \Delta\widetilde{x}^c + \widetilde{x} \circ \Delta\widetilde{s}^c &= -\Delta\widetilde{x}_3 \circ \Delta\widetilde{s}_3. \end{aligned} \tag{12}$$

We get the new iterate $(\widetilde{x}(\alpha), \widetilde{s}(\alpha))$ as follows:

$$(\widetilde{x}(\alpha), \widetilde{s}(\alpha)) = (\widetilde{x} + \alpha\Delta\widetilde{x}_3 + \alpha^2\Delta\widetilde{x}^c, \widetilde{s} + \alpha\Delta\widetilde{s}_3 + \alpha^2\Delta\widetilde{s}^c), \tag{13}$$

where $\alpha \in [0, 1]$ is the step size, which ensures a sufficient reduction in the duality gap and $(\widetilde{x}(\alpha), \widetilde{s}(\alpha)) \in \mathcal{N}(\tau, \beta)$.

The duality gap corresponding to the new iterate is

$$\begin{aligned} \widetilde{\mu}(\alpha) &= \left\langle Q(u)(x + \alpha\Delta x_3 + \alpha^2\Delta x^c), Q(u^{-1})(s + \alpha\Delta s_3 + \alpha^2\Delta s^c)\right\rangle/r \\ &= \left\langle x + \alpha\Delta x_3 + \alpha^2\Delta x^c, s + \alpha\Delta s_3 + \alpha^2\Delta s^c\right\rangle/r = \mu(\alpha). \end{aligned} \tag{14}$$

It is easy to see that

$$\begin{aligned} \rho(\alpha) &= s(\alpha) - \mathcal{A}(x(\alpha)) - q = (s + \alpha\Delta s_3 + \alpha^2\Delta s^c) - \mathcal{A}(x + \alpha\Delta x_3 + \alpha^2\Delta x^c) - q \\ &= (1-\delta\alpha)\rho. \end{aligned} \tag{15}$$

Algorithm 1

Inputs:

Accuracy parameter $\varepsilon > 0$;
neighborhood parameters $0 < \beta \leq 1/2$ and $0 < \tau \leq 1/4$;
the initial solution $(x^0, s^0) \in \mathcal{N}(\tau, \beta)$ with $\mu^0 = \langle x^0, s^0\rangle / r$.

0. Set $k := 0$ and $\chi^k = 1$.
1. If $\chi^k \leq \varepsilon$, then stop; otherwise, Choose a scaling element $u \in \mathcal{C}(x^k, s^k)$ and go to Step 2.
2. Compute the direction $\left(\Delta\widetilde{x}_1^k, \Delta\widetilde{s}_1^k\right)$ by (8) and $\left(\Delta\widetilde{x}_2^k, \Delta\widetilde{s}_2^k\right)$ by (9).
3. Compute the maximum size δ^k, that keeps (10), and calculate the direction$\left(\Delta\widetilde{x}_3^k, \Delta\widetilde{s}_3^k\right)$ by (11).
4. Compute the direction $\left(\Delta\widetilde{x}^{c^k}, \Delta\widetilde{s}^{c^k}\right)$ by (12).
5. Find $\bar{\alpha}^k \in [0, 1]$ such that for all $\alpha_1^k \in [0, \bar{\alpha}^k]$, $\mu(\bar{\alpha}^k) \leq \mu(\alpha_1^k)$.
6. Find $\hat{\alpha}^k \in [0, \bar{\alpha}^k]$ such that for all $\alpha_2^k \in [0, \hat{\alpha}^k]$, $\left(\hat{x}(\alpha_2^k), \hat{s}(\alpha_2^k)\right) \in \mathcal{N}(\tau, \beta)$.
7. Let $(x^{k+1}, s^{k+1}) := \left(Q(u^{-1})\widetilde{x}(\widehat{\alpha}^k), Q(u)\widetilde{s}(\widehat{\alpha}^k)\right)$ and $\mu^{k+1} := \langle x^{k+1}, s^{k+1}\rangle / r$. Set $\phi^{k+1} = \mu^{k+1}/\mu^k$, $\chi^{k+1} = \max\left\{\prod_{i=0}^{k+1}\phi^i, \prod_{i=0}^{k}(1-\delta^i\widehat{\alpha}^i)\right\}$, $k := k+1$ and go to Step 1.

Figure 1. The infeasible-interior-point predictor-corrector algorithm for the Cartesian $P_*(\kappa)$-SCLCP.

Moreover, we have

$$\widetilde{x}(\alpha) \circ \widetilde{s}(\alpha) = T(\alpha) + \Delta\widetilde{x}(\alpha) \circ \Delta\widetilde{s}(\alpha), \tag{16}$$

where,

$$\begin{aligned} T(\alpha) &= \widetilde{x} \circ \widetilde{s} + \alpha\left[(\tau\mu e - \widetilde{x} \circ \widetilde{s})^- + \sqrt{r}\,(\tau\mu e - \widetilde{x} \circ \widetilde{s})^+\right], \\ \Delta\widetilde{x}(\alpha) \circ \Delta\widetilde{s}(\alpha) &= \alpha^3(\Delta\widetilde{x}_3 \circ \Delta\widetilde{s}^c + \Delta\widetilde{s}_3 \circ \Delta\widetilde{x}^c) + \alpha^4(\Delta\widetilde{x}^c \circ \Delta\widetilde{s}^c). \end{aligned} \tag{17}$$

A more formal description of the predictor-corrector algorithm for the Cartesian $P_*(\kappa)$-SCLCP is given in Figure 1.

The following remark is readily verified for Algorithm 1.

Remark 2: Let $\left\{(x^k, s^k)\right\}$ be generated by Algorithm 1. Then for $k \geq 0$, we have

$$s^{k+1} - \mathcal{A}(x^{k+1}) - q = \varphi^{k+1}\left(s^0 - \mathcal{A}(x^0) - q\right),$$

where $\varphi^0 = 1$ and $\varphi^{k+1} = (1 - \delta^k\widehat{\alpha}^k)\varphi^k = \prod_{i=0}^{k}(1 - \delta^i\hat{\alpha}^i) \in [0, 1]$.

From Remark 2, we have

$$\varphi^k = \frac{\left\|s^k - \mathcal{A}(x^k) - q\right\|}{\left\|s^0 - \mathcal{A}(x^0) - q\right\|}.$$

Here φ^k represents the relative infeasibility at (x^k, s^k).

4. Complexity analysis

In this section, we mainly characterize the polynomial complexity of Algorithm 1. In order to achieve the complexity, we list some technical results.

Lemma 4.1 (Lemma 33 in [4]): *Let $v_1, v_2 \in \mathcal{J}$ and G be a positive definite matrix which is symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$. Then*

$$\begin{aligned} \|v_1\| \|v_2\| \leq &\ \sqrt{cond(G)} \left\|G^{-1/2}v_1\right\| \left\|G^{1/2}v_2\right\| \\ \leq &\ \tfrac{1}{2}\sqrt{cond(G)} \left(\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2\right), \end{aligned}$$

where $cond(G) = \lambda_{\max}(G)/\lambda_{\min}(G)$ is the condition number of G.

As the results of Lemma 36 in [4], we present a bound on the condition number of G for some specific search directions in the following lemma.

Lemma 4.2: *For the NT search direction, $cond(G) = 1$. For the xs and sx directions, $cond(G) \leq r/(1-\beta)\tau$.*

Lemma 4.3: *Suppose $x, s, a \in \mathcal{J}$ with $(x, s) \in int\, \mathcal{K} \times int\, \mathcal{K}$, $L(x)L(s) = L(s)L(x)$, $G = L(s)^{-1}L(x)$ and $\mathcal{A}$ has Cartesian $P_*(\kappa)$ property. Then the solution (v_1, v_2) of the following linear system*

$$\begin{aligned} \mathcal{A}(v_1) - v_2 &= 0, \\ L(s)v_1 + L(x)v_2 &= a, \end{aligned}$$

satisfies the following inequality:

$$\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2 \leq (1+2\kappa)\, \|\bar{a}\|^2,$$

where $\bar{a} = \left(L(x)L(s)\right)^{-1/2} a$.

***Proof*:** In the same way as the proof of Lemma 4.2 in [18], we obtain the result. □

Before dealing with the analysis of the algorithm, we recall the following lemma from [21] that will be needed.

Lemma 4.4: *Let $x, s, a, b \in \mathcal{J}$ with $(x, s) \in int\, \mathcal{K} \times int\, \mathcal{K}$, $L(x)L(s) = L(s)L(x)$, $G = L(s)^{-1}L(x)$ and $\mathcal{A}$ has Cartesian $P_*(\kappa)$ property. Then the solution (v_1, v_2) of the following linear system*

$$\begin{aligned} \mathcal{A}(v_1) - v_2 &= b, \\ L(s)v_1 + L(x)v_2 &= a, \end{aligned} \tag{18}$$

satisfies the following inequality:

$$\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2 \leq (1+2\kappa)\left(\|\bar{a}\| + 3\zeta\right)^2,$$

where $\zeta^2 = \inf\left\{\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2 : \mathcal{A}(v_1) - v_2 = b\right\}$.

***Proof*:** By multiplying (18) by $(L(x)L(s))^{-1/2}$, we obtain

$$\begin{aligned} \mathcal{A}(v_1) - v_2 &= b, \\ G^{-1/2}v_1 + G^{1/2}v_2 &= \bar{a}. \end{aligned}$$

Let $(\bar{v}_1, \bar{v}_2) \in \mathcal{J} \times \mathcal{J}$ satisfy equation $\mathcal{A}(\bar{v}_1) - \bar{v}_2 = b$. Hence one has

$$\begin{aligned} \mathcal{A}(v_1 - \bar{v}_1) - (v_2 - \bar{v}_2) &= 0, \\ G^{-1/2}(v_1 - \bar{v}_1) + G^{1/2}(v_2 - \bar{v}_2) &= \bar{a} - \left(G^{-1/2}\bar{v}_1 + G^{1/2}\bar{v}_2\right). \end{aligned} \tag{19}$$

Using (19) and Lemma 4.3, we obtain

$$\begin{aligned} &\sqrt{\left\|G^{-1/2}(v_1 - \bar{v}_1)\right\|^2 + \left\|G^{1/2}(v_2 - \bar{v}_2)\right\|^2} \\ &\quad \le \sqrt{1+2\kappa}\,\left\|\bar{a} - \left(G^{-1/2}\bar{v}_1 + G^{1/2}\bar{v}_2\right)\right\| \le \sqrt{1+2\kappa}\,\left(\|\bar{a}\| + \left\|G^{-1/2}\bar{v}_1 + G^{1/2}\bar{v}_2\right\|\right) \\ &\quad \le \sqrt{1+2\kappa}\left(\|\bar{a}\| + \sqrt{\left\|G^{-1/2}\bar{v}_1\right\|^2 + \left\|G^{1/2}\bar{v}_2\right\|^2 + 2\left\|G^{-1/2}\bar{v}_1\right\|\left\|G^{1/2}\bar{v}_2\right\|}\right) \\ &\quad \le \sqrt{1+2\kappa}\left(\|\bar{a}\| + \sqrt{2}\sqrt{\left\|G^{-1/2}\bar{v}_1\right\|^2 + \left\|G^{1/2}\bar{v}_2\right\|^2}\right). \end{aligned} \tag{20}$$

On the other hand, by (20), we have

$$\begin{aligned} &\sqrt{\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2} \\ &\quad \le \sqrt{\left\|G^{-1/2}(v_1 - \bar{v}_1)\right\|^2 + \left\|G^{1/2}(v_2 - \bar{v}_2)\right\|^2} + \sqrt{\left\|G^{-1/2}\bar{v}_1\right\|^2 + \left\|G^{1/2}\bar{v}_2\right\|^2} \\ &\quad \le \sqrt{1+2\kappa}\,\|\bar{a}\| + \left(1 + \sqrt{2+4\kappa}\right)\sqrt{\left\|G^{-1/2}\bar{v}_1\right\|^2 + \left\|G^{1/2}\bar{v}_2\right\|^2} \\ &\quad \le \sqrt{1+2\kappa}\left(\|\bar{a}\| + 3\sqrt{\left\|G^{-1/2}\bar{v}_1\right\|^2 + \left\|G^{1/2}\bar{v}_2\right\|^2}\right). \end{aligned} \tag{21}$$

Therefore, by (21), we have

$$\sqrt{\left\|G^{-1/2}v_1\right\|^2 + \left\|G^{1/2}v_2\right\|^2} \le \sqrt{1+2\kappa}\left(\|\bar{a}\| + 3\zeta\right),$$

which completes the proof. □

Since the proof techniques of the following lemma are the same as in Lemma 4.6 in [21], we will only present it without proof.

Lemma 4.5: *Let $G = L(\widetilde{s})^{-1}L(\widetilde{x})$, $(\widetilde{x}, \widetilde{s})$ and $(\Delta\widetilde{x}_1, \Delta\widetilde{s}_1)$ generated by Algorithm 1. Then we have*

$$\inf\left\{\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|^2 + \left\|G^{1/2}\Delta\widetilde{s}_1\right\|^2 : \widetilde{\mathcal{A}}(\Delta\widetilde{x}_1) - \Delta\widetilde{s}_1 = \widetilde{\rho}\right\} \le 4(1+4\kappa)^2\frac{r^2\mu}{(1-\beta)\,\tau}.$$

Lemma 4.6 (Lemma 5.2 in [15]): *Let $(\widetilde{x}, \widetilde{s}) \in \mathcal{N}(\tau, \beta)$. Then*

$$\mathbf{Tr}(\tau\mu e - \widetilde{x} \circ \widetilde{s})^+ \le \sqrt{r}\beta\tau\mu.$$

Lemma 4.7 (Lemma 5.3 in [15]): *Let $(\widetilde{x}, \widetilde{s})$ generated by Algorithm 1. Then, we have*

$$\left\|(L(\widetilde{x})L(\widetilde{s}))^{-1/2}\left[(\tau\mu e - \widetilde{x} \circ \widetilde{s})^- + \sqrt{r}\,(\tau\mu e - \widetilde{x} \circ \widetilde{s})^+\right]\right\|^2 \le (1+\beta\tau)r\mu.$$

Lemma 4.8: *Let $G = L(\widetilde{s})^{-1}L(\widetilde{x})$, $(\widetilde{x}, \widetilde{s})$ and $(\Delta\widetilde{x}_1, \Delta\widetilde{s}_1)$ generated by Algorithm 1. Then, there exists a constant $\omega \ge 12$, such that*

$$\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|^2 + \left\|G^{1/2}\Delta\widetilde{s}_1\right\|^2 \le \omega^2(1+2\kappa)^3 r^2\mu.$$

***Proof*:** Applying Lemma 4.4 to the system (8), Lemmas 4.5 and 4.7, we obtain

$$\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}_1\right\|^2\leq(1+2\kappa)\left(\sqrt{(1+\beta\tau)\mu r}+6(1+4\kappa)r\sqrt{\frac{\mu}{(1-\beta)\,\tau}}\right)^2$$
$$\leq(1+2\kappa)\left(1+6(1+4\kappa)\right)^2\eta^2r^2\mu\leq(1+2\kappa)\left(12(1+2\kappa)\right)^2\eta^2r^2\mu,$$

where, $\eta=\max\left\{\sqrt{(1+\beta\tau)},\sqrt{\frac{1}{(1-\beta)\tau}}\right\}\geq 1$. By $\omega:=12\eta$, the proof is completed. □

Lemma 4.9: *Let $G=L(\widetilde{s})^{-1}L(\widetilde{x})$. Then we have*

$$\begin{aligned}
(1):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_1\right)\right|\leq\tfrac{1}{2}\omega^2(1+2\kappa)^3r^2\mu,\\
(2):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}_2\circ\Delta\widetilde{s}_2\right)\right|\leq\tfrac{1}{2}(1+2\kappa)(1+\beta\tau)r\mu,\\
(3):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_2\right)\right|\leq\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu,\\
(4):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}_2\circ\Delta\widetilde{s}_1\right)\right|\leq\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu.
\end{aligned}$$

***Proof*:** Using Lemma 4.8, we obtain

$$\begin{aligned}
\left|\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_1\right)\right|&=\left|\mathbf{Tr}\left(\left(G^{-1/2}\Delta\widetilde{x}_1\right)\circ\left(G^{1/2}\Delta\widetilde{s}_1\right)\right)\right|\\
&\leq\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|\left\|G^{1/2}\Delta\widetilde{s}_1\right\|\leq\frac{1}{2}\left(\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}_1\right\|^2\right)\\
&\leq\frac{1}{2}\omega^2(1+2\kappa)^3r^2\mu,
\end{aligned}\tag{22}$$

which implies the first part of the lemma. From Lemmas 4.3 and 4.7, we have

$$\left\|G^{-1/2}\Delta\widetilde{x}_2\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}_2\right\|^2\leq(1+2\kappa)(1+\beta\tau)r\mu.\tag{23}$$

Similar to the proof of (22) and using (23), we can obtain the second part.

By Lemma 4.8 and (23), we derive

$$\begin{aligned}
\left|\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_2\right)\right|&=\left|\mathbf{Tr}\left(\left(G^{-1/2}\Delta\widetilde{x}_1\right)\circ\left(G^{1/2}\Delta\widetilde{s}_2\right)\right)\right|\\
&\leq\left\|G^{-1/2}\Delta\widetilde{x}_1\right\|\left\|G^{1/2}\Delta\widetilde{s}_2\right\|\leq\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu,
\end{aligned}$$

which implies the third part. Similarly, we also obtain

$$\left|\mathbf{Tr}\left(\Delta\widetilde{x}_2\circ\Delta\widetilde{s}_1\right)\right|\leq\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu,$$

which follows the forth part. The proof is completed. □

Lemma 4.10: *Let $G=L(\widetilde{s})^{-1}L(\widetilde{x})$. Then we have*

$$\begin{aligned}
(a):&\ \left\|G^{-1/2}\Delta\widetilde{x}_3\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}_3\right\|^2\leq\tfrac{11}{5}(1+2\kappa)(1+\beta\tau)r\mu,\\
(b):&\ \left\|\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right\|\leq\frac{121\left(cond(G)\right)^{3/2}(1+2\kappa)^3(1+\beta\tau)^2r^2\mu}{200(1-\beta)\tau},\\
(c):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right)\right|\leq\frac{121(1+2\kappa)^3cond(G)(1+\beta\tau)^2r^2\mu}{200(1-\beta)\tau},\\
(d):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}^c\right)\right|\leq\frac{9\sqrt{cond(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}},\\
(e):&\ \left|\mathbf{Tr}\left(\Delta\widetilde{s}_3\circ\Delta\widetilde{x}^c\right)\right|\leq\frac{9\sqrt{cond(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}}.
\end{aligned}$$

***Proof*:** Using Lemma 4.7 and (10), we have

$$\begin{aligned}\left\|G^{-1/2}\Delta\widetilde{x}_3\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}_3\right\|^2&=\left\|G^{-1/2}\Delta\widetilde{x}_3+G^{1/2}\Delta\widetilde{s}_3\right\|^2-2\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}_3\right)\\&\le(1+\beta\tau)r\mu+\tfrac{6}{5}(1+2\kappa)(1+\beta\tau)r\mu\le\tfrac{11}{5}(1+2\kappa)(1+\beta\tau)r\mu,\end{aligned}$$

which follows the inequality (a). From the first part and Lemma 4.1, we derive

$$\|\Delta\widetilde{x}_3\circ\Delta\widetilde{s}_3\|\le\tfrac{11}{10}\sqrt{\mathrm{cond}(G)}(1+2\kappa)(1+\beta\tau)r\mu.$$

In the same way as the proof of Lemma 2 in [27], from Lemma 4.3 and the previous inequality, we obtain

$$\begin{aligned}\left\|G^{-1/2}\Delta\widetilde{x}^c\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}^c\right\|^2&\le(1+2\kappa)\left\|G^{-1/2}\Delta\widetilde{x}^c+G^{1/2}\Delta\widetilde{s}^c\right\|^2\\&=(1+2\kappa)\left\|\left(L(\widetilde{x})L(\widetilde{s})\right)^{-1/2}\left(-\Delta\widetilde{x}_3\circ\Delta\widetilde{s}_3\right)\right\|^2\le\frac{(1+2\kappa)}{(1-\beta)\tau\mu}\|\Delta\widetilde{x}_3\circ\Delta\widetilde{s}_3\|^2\\&\le\frac{121(1+2\kappa)^3\mathrm{cond}(G)(1+\beta\tau)^2r^2\mu}{100(1-\beta)\tau}.\end{aligned}\tag{24}$$

Using Lemma 4.1 in (24), the inequality (b) is obtained.

Since $\left|\mathbf{Tr}\left(\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right)\right|\le\frac{1}{2}\left(\left\|G^{-1/2}\Delta\widetilde{x}^c\right\|^2+\left\|G^{1/2}\Delta\widetilde{s}^c\right\|^2\right)$, it follows that

$$\left|\mathbf{Tr}\left(\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right)\right|\le\frac{121(1+2\kappa)^3\mathrm{cond}(G)(1+\beta\tau)^2r^2\mu}{200(1-\beta)\tau}.$$

Using (a) and (24), it readily follows that

$$\begin{aligned}\left|\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}^c\right)\right|&=\left|\mathbf{Tr}\left(\left(G^{-1/2}\Delta\widetilde{x}_3\right)\circ\left(G^{1/2}\Delta\widetilde{s}^c\right)\right)\right|\\&\le\left\|G^{-1/2}\Delta\widetilde{x}_3\right\|\left\|G^{1/2}\Delta\widetilde{s}^c\right\|\le\frac{9\sqrt{\mathrm{cond}(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}}.\end{aligned}$$

Similarly, we have

$$\left|\mathbf{Tr}\left(\Delta\widetilde{s}_3\circ\Delta\widetilde{x}^c\right)\right|\le\frac{9\sqrt{\mathrm{cond}(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}}.$$

This completes the proof. □

From (24), Lemmas 4.1 and 4.10, we have

$$\begin{aligned}\|\Delta\widetilde{x}_3\|\,\|\Delta\widetilde{s}^c\|&\le\frac{9(1+2\kappa)^2\mathrm{cond}(G)(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}},\\\|\Delta\widetilde{s}_3\|\,\|\Delta\widetilde{x}^c\|&\le\frac{9(1+2\kappa)^2\mathrm{cond}(G)(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}}.\end{aligned}\tag{25}$$

Lemma 4.11: *The maximum size* δ*, that keeps* (10)*, satisfies*

$$\delta\ge\frac{1}{25\omega(1+2\kappa)r^{1/2}}:=\delta_0.$$

***Proof*:** Using Lemma 4.9, we have

$$\begin{aligned}
&\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}_3\right)+\frac{3}{5}(1+2\kappa)(1+\beta\tau)r\mu=\delta^2\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_1\right)+(1-\delta)^2\mathbf{Tr}\left(\Delta\widetilde{x}_2\circ\Delta\widetilde{s}_2\right)\\
&\quad+\delta(1-\delta)\mathbf{Tr}\left(\Delta\widetilde{x}_1\circ\Delta\widetilde{s}_2+\Delta\widetilde{s}_1\circ\Delta\widetilde{x}_2\right)+\frac{3}{5}(1+2\kappa)(1+\beta\tau)r\mu\\
&\geq\delta^2\left(-\frac{1}{2}\omega^2(1+2\kappa)^3r^2\mu\right)+(1-\delta)^2\left(-\frac{1}{2}(1+2\kappa)(1+\beta\tau)r\mu\right)\\
&\quad+\delta(1-\delta)\left(-2\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu\right)+\frac{3}{5}(1+2\kappa)(1+\beta\tau)r\mu\\
&\geq\delta^2\left(-\frac{1}{2}\omega^2(1+2\kappa)^3r^2\mu\right)-\frac{1}{2}(1+2\kappa)(1+\beta\tau)r\mu\\
&\quad-2\delta\omega\sqrt{1+\beta\tau}(1+2\kappa)^2r^{3/2}\mu+\frac{3}{5}(1+2\kappa)(1+\beta\tau)r\mu\\
&=-(1+2\kappa)r\mu\left[\frac{1}{2}\delta^2\omega^2(1+2\kappa)^2r+2\delta\omega\sqrt{1+\beta\tau}(1+2\kappa)r^{1/2}-\frac{1}{10}(1+\beta\tau)\right]\\
&:=-(1+2\kappa)r\mu f(\delta).
\end{aligned}$$

In order to find a lower bound for δ satisfying (10), it suffices to obtain δ such that $f(\delta)\leq 0$. The quadratic equation $f(\delta)=0$ has a unique positive root

$$\begin{aligned}
\hat{\delta}&=\frac{-2\omega\sqrt{1+\beta\tau}(1+2\kappa)r^{1/2}+\sqrt{\frac{21}{5}}\omega\sqrt{(1+\beta\tau)}(1+2\kappa)r^{1/2}}{\omega^2(1+2\kappa)^2r}\\
&\geq\frac{1}{25\omega(1+2\kappa)r^{1/2}},
\end{aligned}$$

which follows the desired result. □

From (16), it is obvious that

$$\widetilde{\mu}(\alpha)=\mu(\alpha)=\mu+\alpha\left[(\tau-1)\mu+\frac{\sqrt{r}-1}{r}\mathbf{Tr}(\tau\mu e-\widetilde{x}\circ\widetilde{s})^+\right]+\frac{1}{r}\mathbf{Tr}(\Delta\widetilde{x}(\alpha)\circ\Delta\widetilde{s}(\alpha)). \tag{26}$$

Lemma 4.12: *Let $\beta\leq 1/2$ and $\tau\leq 1/4$. Then the maximum step size α such that $\mu(\alpha)$ decreases in $[0,\alpha]$, satisfies*

$$\alpha\geq\frac{\left((1-\beta)\tau\right)^{1/3}\sqrt{1-\beta\tau-\tau}}{5\left(cond(G)\right)^{1/3}(1+2\kappa)(1+\beta\tau)^{3/4}r^{1/3}}:=\bar{\alpha}_0.$$

***Proof*:** From (17), (26) and Lemma 4.6, we obtain

$$\begin{aligned}
\mu'(\alpha)&=(\tau-1)\mu+\frac{\sqrt{r}-1}{r}\mathbf{Tr}(\tau\mu e-\widetilde{x}\circ\widetilde{s})^++\frac{1}{r}\mathbf{Tr}'(\Delta\widetilde{x}(\alpha)\circ\Delta\widetilde{s}(\alpha))\\
&\leq\tau\mu-\mu+\frac{\sqrt{r}-1}{r}\sqrt{r}\beta\tau\mu+\frac{3\alpha^2}{r}\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}^c+\Delta\widetilde{s}_3\circ\Delta\widetilde{x}^c\right)+\frac{4\alpha^3}{r}\mathbf{Tr}\left(\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right)\\
&\leq\left(\tau+\beta\tau-1\right)\mu+\frac{3\alpha^2}{r}\mathbf{Tr}\left(\Delta\widetilde{x}_3\circ\Delta\widetilde{s}^c+\Delta\widetilde{s}_3\circ\Delta\widetilde{x}^c\right)+\frac{4\alpha^3}{r}\mathbf{Tr}\left(\Delta\widetilde{x}^c\circ\Delta\widetilde{s}^c\right)\\
&\leq\Big[\left(\tau+\beta\tau-1\right)+\alpha^2\frac{54\sqrt{\mathrm{cond}(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2}r^{1/2}}{5\sqrt{(1-\beta)\tau}}\\
&\quad+\alpha^3\frac{121(1+2\kappa)^3\mathrm{cond}(G)(1+\beta\tau)^2r}{50(1-\beta)\tau}\Big]\mu,
\end{aligned}\tag{27}$$

where the third inequality follows from Lemma 4.10. Let $f_1(\alpha)$ be defined as follows:

$$f_1(\alpha) := \big(\tau + \beta\tau - 1\big) + \alpha^2 \frac{54\sqrt{\mathrm{cond}(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2} r^{1/2}}{5\sqrt{(1-\beta)\tau}} + \alpha^3 \frac{121(1+2\kappa)^3 \mathrm{cond}(G)(1+\beta\tau)^2 r}{50(1-\beta)\tau}.$$

Therefore,

$$f_1(\bar{\alpha}_0) = \big(\tau + \beta\tau - 1\big) + \frac{54\big((1-\beta)\tau\big)^{1/6}(1-\beta\tau-\tau)}{125\big(\mathrm{cond}(G)\big)^{1/6} r^{1/6}} + \frac{121(1-\beta\tau-\tau)^{3/2}}{6250(1+\beta\tau)^{1/4}} \leq 0,$$

which implies that for all $\alpha \in [0, \bar{\alpha}_0]$, $\mu'(\alpha) \leq 0$. This completes the proof. □

In the following, we give a sufficient condition to keep all the iterates in the neighbourhood $\mathcal{N}(\tau, \beta)$. In order to keep the iterates in this neighbourhood, we need the following lemma.

Lemma 4.13: *Let* $0 < \mu(\alpha)$ *decreases in* $[0, \alpha]$. *Then we have*

$$\begin{cases} \left\| \big(\tau\mu(\alpha)e - T(\alpha)\big)^+ \right\| \leq \big(1 - \alpha\sqrt{r}\big)\,\beta\tau\mu(\alpha) & \text{if } \alpha < 1/\sqrt{r}, \\ \left\| \big(\tau\mu(\alpha)e - T(\alpha)\big)^+ \right\| = 0 & \text{if } \alpha \geq 1/\sqrt{r}. \end{cases}$$

***Proof*:** The proof is similar to the proof of Lemma 5.8 in [15], therefore it is omitted. □

Lemma 4.14: *Let* $\beta \leq 1/2,\ \tau \leq 1/4$ *and* $\hat{\alpha}$ *be as defined in Step 6 of Algorithm 1. Then*

$$\hat{\alpha} \geq \frac{\sqrt{\beta\tau}\,\big((1-\beta)\tau\big)^{1/3}}{5(1+2\kappa)\sqrt{cond(G)}(1+\beta\tau)^{3/4} r^{1/2}} := \hat{\alpha}_0.$$

***Proof*:** If $\hat{\alpha} \geq 1/\sqrt{r}$, then we have $\hat{\alpha} \geq \hat{\alpha}^0$, which follows the lemma. Thus, we will restrict ourselves to the case where $\hat{\alpha} < 1/\sqrt{r}$.

Using (17), (25), Lemmas 2.6, 4.10 and 4.13, we have

$$\begin{aligned} \left\| \big(\tau\mu(\alpha)e - \widetilde{x}(\alpha) \circ \widetilde{s}(\alpha)\big)^+ \right\| &\leq \left\| \big(\tau\mu(\alpha)e - T(\alpha)\big)^+ \right\| + \left\| \big(\Delta\widetilde{x}(\alpha) \circ \Delta\widetilde{s}(\alpha)\big)^- \right\| \\ &\leq \big(1 - \alpha\sqrt{r}\big)\,\beta\tau\mu(\alpha) + \alpha^3 \frac{18(1+2\kappa)^2 \mathrm{cond}(G)(1+\beta\tau)^{3/2} r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}} \\ &\quad + \alpha^4 \frac{121\,\big(\mathrm{cond}(G)\big)^{3/2}\,(1+2\kappa)^3(1+\beta\tau)^2 r^2\mu}{200(1-\beta)\tau}. \end{aligned} \tag{28}$$

Using (26) and Lemma 4.10, we have

$$\begin{aligned} \mu(\alpha) &\geq \mu + \alpha(\tau-1)\mu + \frac{1}{r}\mathbf{Tr}\big(\Delta\widetilde{x}(\alpha) \circ \Delta\widetilde{s}(\alpha)\big) \\ &\geq \mu + \alpha(\tau-1)\mu - \frac{\alpha^3}{r}\Big(\big|\mathbf{Tr}\big(\Delta\widetilde{x}_3 \circ \Delta\widetilde{s}^c\big)\big| + \big|\mathbf{Tr}\big(\Delta\widetilde{s}_3 \circ \Delta\widetilde{x}^c\big)\big|\Big) - \frac{\alpha^4}{r}\big|\mathbf{Tr}\big(\Delta\widetilde{x}^c \circ \Delta\widetilde{s}^c\big)\big| \\ &\geq \big[1 + \alpha(\tau-1)\big]\mu - \alpha^3 \frac{18(1+2\kappa)^2\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/2} r^{1/2}\mu}{5\sqrt{(1-\beta)\tau}} \\ &\quad - \alpha^4 \frac{121\,\big(\mathrm{cond}(G)\big)\,(1+2\kappa)^3(1+\beta\tau)^2 r\mu}{200(1-\beta)\tau}. \end{aligned} \tag{29}$$

From (28) and (29), we obtain

$$\begin{aligned}
&\left\|\left(\tau\mu(\alpha)e-\widetilde{x}(\alpha)\circ\widetilde{s}(\alpha)\right)^{+}\right\|-\beta\tau\mu(\alpha)\leq\alpha^{3}\frac{18(1+2\kappa)^{2}\mathrm{cond}(G)(1+\beta\tau)^{3/2}r^{3/2}\mu}{5\sqrt{(1-\beta)\tau}}\\
&\quad+\alpha^{4}\frac{121\left(\mathrm{cond}(G)\right)^{3/2}(1+2\kappa)^{3}(1+\beta\tau)^{2}r^{2}\mu}{200(1-\beta)\tau}-\alpha\sqrt{r}\beta\tau\mu(\alpha)\\
&\leq\alpha\sqrt{r}\beta\tau\mu\Bigg(\alpha^{2}\frac{18(1+2\kappa)^{2}\mathrm{cond}(G)(1+\beta\tau)^{3/2}r}{5\beta\tau\sqrt{(1-\beta)\tau}}\\
&\quad+\alpha^{3}\frac{121\left(\mathrm{cond}(G)\right)^{3/2}(1+2\kappa)^{3}(1+\beta\tau)^{2}r^{3/2}}{200\beta\tau(1-\beta)\tau}\\
&\quad+\alpha^{3}\frac{18(1+2\kappa)^{2}\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/2}r^{1/2}}{5\sqrt{(1-\beta)\tau}}\\
&\quad+\alpha^{4}\frac{121\left(\mathrm{cond}(G)\right)(1+2\kappa)^{3}(1+\beta\tau)^{2}r}{200(1-\beta)\tau}+\alpha(1-\tau)-1\Bigg):=\alpha\sqrt{r}\beta\tau\mu f_{2}(\alpha). \qquad (30)
\end{aligned}$$

Thus, it is easily concluded that

$$\begin{aligned}
f_{2}(\hat{\alpha}_{0})&=\frac{18\left((1-\beta)\tau\right)^{1/6}}{125}+\frac{121\sqrt{\beta\tau}}{25000(1+\beta\tau)^{1/4}}\\
&\quad+\frac{18\left(\beta\tau\right)^{3/2}\sqrt{(1-\beta)\tau}}{625(1+2\kappa)\mathrm{cond}(G)(1+\beta\tau)^{3/4}r}+\frac{121\left(\beta\tau\right)^{2}\left((1-\beta)\tau\right)^{1/3}}{125000(1+2\kappa)\mathrm{cond}(G)(1+\beta\tau)r}\\
&\quad+\frac{\sqrt{\beta\tau}\left((1-\beta)\tau\right)^{1/3}(1-\tau)}{5(1+2\kappa)\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/4}r^{1/2}}-1\leq 0. \qquad (31)
\end{aligned}$$

From (30) and (31), we have for all $\alpha\in[0,\,\hat{\alpha}^{0}]$,

$$\left\|\left(\tau\mu(\alpha)e-\widetilde{x}(\alpha)\circ\widetilde{s}(\alpha)\right)^{+}\right\|\leq\beta\tau\mu(\alpha).$$

Then, by Lemma 2.5, we have $\det(\widetilde{x}(\alpha))\neq 0$ and $\det(\widetilde{s}(\alpha))\neq 0$ for all $\alpha\in[0,\,\hat{\alpha}^{0}]$. Since $\det(\widetilde{x}(0))=\det(\widetilde{x})>0$ and $\det(\widetilde{s}(0))=\det(\widetilde{s})>0$, by continuity, it follows that in this interval $\widetilde{x}(\alpha)\in\mathrm{int}\,\mathcal{K}$ and $\widetilde{s}(\alpha)\in\mathrm{int}\,\mathcal{K}$. On the other hand, since $\beta\leq 1/2$ and $\tau\leq 1/4$, we have $\hat{\alpha}_0\leq\bar{\alpha}_0$. This completes the proof of the lemma. □

4.1. Polynomial complexity

In this subsection, we present the polynomial complexity for Algorithm 1.

Theorem 4.15: *The Algorithm 1 terminates in*

$$O\left(\sqrt{cond(G)}(1+\kappa)^{2}r\log\varepsilon^{-1}\right)$$

iterations with (x^k, s^k) *such that* $\left\|s^k-\mathcal{A}(x^k)-q\right\|\leq\varepsilon\left\|s^0-\mathcal{A}(x^0)-q\right\|$ *and* $\langle x^k, s^k\rangle\leq\varepsilon\langle x^0, s^0\rangle$.

Proof: In the same way as the proof of (27), using Lemmas 4.10 and 4.12, we can easily verify that

$$
\begin{aligned}
\mu(\hat{\alpha}) &\leq \mu(\hat{\alpha}_0) \\
&= \mu + \hat{\alpha}_0 \left[(\tau - 1)\mu + \frac{\sqrt{r}-1}{r} \mathbf{Tr}\left((\tau\mu e - \widetilde{x} \circ \widetilde{s})^+ \right) \right] + \frac{1}{r} \mathbf{Tr}\left(\Delta\widetilde{x}(\hat{\alpha}_0) \circ \Delta\widetilde{s}(\hat{\alpha}_0) \right) \\
&\leq \mu + \hat{\alpha}_0 \left[(\tau - 1)\mu + \frac{\sqrt{r}-1}{r} \mathbf{Tr}\left((\tau\mu e - \widetilde{x} \circ \widetilde{s})^+ \right) \right] \\
&\quad + \frac{\hat{\alpha}_0^3}{r} \left(\left| \mathbf{Tr}\left(\Delta\widetilde{x}_3 \circ \Delta\widetilde{s}^c \right) \right| + \left| \mathbf{Tr}\left(\Delta\widetilde{s}_3 \circ \Delta\widetilde{x}^c \right) \right| \right) + \frac{\hat{\alpha}_0^4}{r} \left| \mathbf{Tr}\left(\Delta\widetilde{x}^c \circ \Delta\widetilde{s}^c \right) \right| \\
&\leq \big[1 - \hat{\alpha}_0 \big(1 - \tau - \beta\tau - \hat{\alpha}_0^2 \frac{18\sqrt{\mathrm{cond}(G)}(1+2\kappa)^2(1+\beta\tau)^{3/2} r^{1/2}}{5\sqrt{(1-\beta)\tau}} \\
&\quad - \hat{\alpha}_0^3 \frac{121(1+2\kappa)^3 \mathrm{cond}(G)(1+\beta\tau)^2 r}{200(1-\beta)\tau} \big) \big] \mu.
\end{aligned} \tag{32}
$$

Substituting $\hat{\alpha}_0 = \frac{\sqrt{\beta\tau}\left((1-\beta)\tau\right)^{1/3}}{5(1+2\kappa)\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/4} r^{1/2}}$ into (32), we have

$$
\begin{aligned}
\mu(\hat{\alpha}) &\leq \left[1 - \hat{\alpha}_0 \left(1 - \tau - \beta\tau - \frac{18\left(\beta\tau\right)\left((1-\beta)\tau\right)^{1/6}}{125\sqrt{\mathrm{cond}(G)} r^{1/2}} - \frac{121\left(\beta\tau\right)^{3/2}}{25000\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{1/4} r^{1/2}} \right) \right] \mu \\
&\leq \left[1 - \hat{\alpha}_0 \left(1 - \tau - \beta\tau - \frac{18}{125} - \frac{121}{25000} \right) \right] \mu \leq \left[1 - \hat{\alpha}_0 \left(\frac{21279}{25000} - \tau - \beta\tau \right) \right] \mu \\
&\leq \left[1 - \frac{\theta\sqrt{\beta\tau}\left((1-\beta)\tau\right)^{1/3}}{10(1+\kappa)\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/4} r^{1/2}} \right] \mu,
\end{aligned}
$$

where $\theta = \left(\frac{21279}{25000} - \tau - \beta\tau\right)$.

Thus, the inequality $\mu(\hat{\alpha}) \leq \varepsilon\mu^0$ holds if

$$
\left(1 - \frac{\theta\sqrt{\beta\tau}\left((1-\beta)\tau\right)^{1/3}}{10(1+\kappa)\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/4} r^{1/2}} \right)^k \leq \varepsilon\mu^0. \tag{33}
$$

It is easy to verify that if $k \geq \frac{10(1+\kappa)\sqrt{\mathrm{cond}(G)}(1+\beta\tau)^{3/4} r^{1/2} \log\varepsilon^{-1}}{\theta\sqrt{\beta\tau}\left((1-\beta)\tau\right)^{1/3}}$, then (33) holds.

Let $\delta_0 = \frac{1}{25\omega(1+2\kappa)r^{1/2}}$, from Remark 2, we have

$$
\varphi^k = \frac{\|s^k - \mathcal{A}(x^k) - q\|}{\|s^0 - \mathcal{A}(x^0) - q\|} = \prod_{i=0}^{k} (1 - \delta^i \hat{\alpha}^i) \leq \left(1 - \delta_0\hat{\alpha}_0\right)^k,
$$

which implies that $\varphi^k \leq \varepsilon$ when $k \geq \frac{\log\varepsilon^{-1}}{\delta_0\hat{\alpha}_0}$.

The desired result immediately follows from the above inequality. □

To obtain complexity of the algorithm for the NT search direction and the *xs* and *sx* search directions, we use Theorem 4.15 and Lemma 4.2.

Corollary 4.16: *If the NT search direction is used, the iteration complexity of Algorithm 1 is* $O\left((1+\kappa)^2 r \log\varepsilon^{-1}\right)$. *If the xs and sx search directions are used, the iteration complexities of Algorithm 1 are* $O((1+\kappa)^2 r^{3/2} \log\varepsilon^{-1})$.

5. Concluding remarks

In this paper, we have presented and analysed a predictor-corrector infeasible-IPM based on a wide neighbourhood for the Cartesian $P_*(\kappa)$-SCLCP. Using the theory of Euclidean Jordan algebras and some elegant tools, we proved the convergence of the algorithm for a commutative class of search directions that coincides with the currently best-known theoretical complexity bounds for infeasible-IPMs for the Cartesian $P_*(\kappa)$-SCLCP. Compared with the results in [19,21], the complexity bound is reduced by a factor of r.

Acknowledgements

The authors would like to thank the anonymous referees for their useful comments and suggestions, which helped to improve the presentation of this paper. The authors would like to thank Shahrekord University for financial support. The authors were also partially supported by the Center of Excellence for Mathematics, University of Shahrekord, Shahrekord, Iran. The second and third authors wish to thank the York University, Professor Michael Chen and his group for hospitality during their recent sabbatical.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Shahrekord University [grant numbers 94GRD1M2003, 94GRD1M1034].

References

[1] Karmarkar NK. A new polynomial-time algorithm for linear programming. Combinatorica. 1984;4:373–395.
[2] Roos C, Terlaky T, Vial J-Ph. Theory and algorithms for linear optimization. An interior-point approach. New York (NY): Wiley; 1997.
[3] Mansouri H, Roos C. Simplified OnL infeasible interior-point algorithm for linear optimization using full-Newton steps. Optim. Methods Softw. 2007;22:519–530.
[4] Schmieta SH, Alizadeh F. Extension of primal-dual interior-point algorithms to symmetric cones. Math. Program. 2003;96:409–438.
[5] Wang GQ, Bai YQ. A new primal-dual path-following interior-point algorithm for semidefinite optimization. J. Math. Anal. Appl. 2009;353:339–349.
[6] Bai YQ, Wang GQ, Roos C. Primal-dual interior-point algorithms for second-order cone optimization based on kernel functions. Nonlinear Anal. 2009;70:3584–3602.
[7] Gu G, Zangiabadi M, Roos C. Full Nesterov–Todd step interior-point methods for symmetric optimization. Eur. J. Oper. Res. 2011;214:473–484.
[8] Zangiabadi M, Gu G, Roos C. A full Nesterov–Todd step infeasible interior-point method for second-order cone optimization. J. Optim. Theory Appl. 2013;158:816–858.
[9] Nesterov YE, Nemirovski A. Interior point polynomial algorithms in convex programming. Philadelphia: SIAM; 1994.
[10] Nesterov YE, Todd MJ. Primal-dual interior-point methods for self-scaled cones. SIAM J. Optim. 1998;8:324–364.
[11] Nesterov YE, Todd MJ. Self-scaled barriers and interior-point methods for convex programming. Math. Oper. Res. 1997;22:1–42.
[12] Ai W. Neighborhood-following algorithms for linear programming. Sci. China Ser. A. 2004;47:812–820.
[13] Ai W, Zhang S. An $O(\sqrt{n}L)$ iteration primal.dual path-following method, based on wide neighborhoods and large updates, for monotone LCP. SIAM J. Optim. 2005;16:400–417.
[14] Li Y, Terlaky T. A new class of large neighborhood path-following interior point algorithms for semidefinite optimization with $O(\sqrt{n}\log(Tr(X^0S^0)/\varepsilon))$ iteration complexity. SIAM J. Optim. 2010;20:2853–2875.
[15] Liu H, Yang X, Liu C. A new wide neighborhood primal-dual infeasible-interior-point method for symmetric cone programming. J. Optim. Theory Appl. 2013;158:796–815.
[16] Yang X, Liu H, Zhang Y. A new strategy in the complexity analysis of an infeasible-interior-point method for symmetric cone programming. J. Optim. Theory Appl. 2015;166:572–587.
[17] Kojima M, Megiddo N, Noma T, et al. A unified approach to interior point algorithms for linear complementarity problems. Vol. 538, Lecture notes in computer science. New York (NY): Springer; 1991.

[18] Luo ZY, Xiu NH. Path-following interior point algorithms for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Sci. China Ser. A. 2009;52:1769–784.
[19] Liu X, Liu H, Liu C. Infeasible Mehrotra-type predictor-corrector interior-point algorithm for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Numer. Funct. Anal. Optim. 2014;35:588–610.
[20] Zangiabadi M, Sayadi Shahraki M, Mansouri H. A large-update interior-point method for Cartesian $P_*(\kappa)$-LCP over symmetric cones. J. Math. Model Algorithm. 2014;13:537–556.
[21] Liu X, Liu H, Wang W. Polynomial convergence of Mehrotra-type predictor-corrector algorithm for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Optimization. 2015;64:815–837.
[22] Sayadi Shahraki M, Zangiabadi M, Mansouri H. A wide neighborhood interior-point method for Cartesian $P_*(\kappa)$-LCP over symmetric cones. J. Oper. Res. Soc. China. 2015;3:331–345.
[23] Wang GQ, Lesaja G. Full Nesterov–Todd step feasible interior-point method for the Cartesian $P_*(\kappa)$-SCLCP. Optim. Methods Softw. 2013;28:600–618.
[24] Wang GQ, Bai YQ. A class of polynomial interior-point algorithms for the Cartesian P-matrix linear complementarity problem over symmetric cones. J. Optim. Theory Appl. 2012;152:739–772.
[25] Faraut J, Korányi A. Analysis on symmetric cones. New York (NY): Oxford University Press; 1994.
[26] Liu C. Study on complexity of some interior-point algorithms in conic programming [PhD thesis]. Xi'an: Xidian University; 2012 (in Chinese).
[27] Zhang J, Zhang K. Polynomial complexity of an interior point algorithm with a second order corrector step for symmetric cone programming. Math. Methods Oper. Res. 2011;73:75–90.