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Neda Ahanjideh

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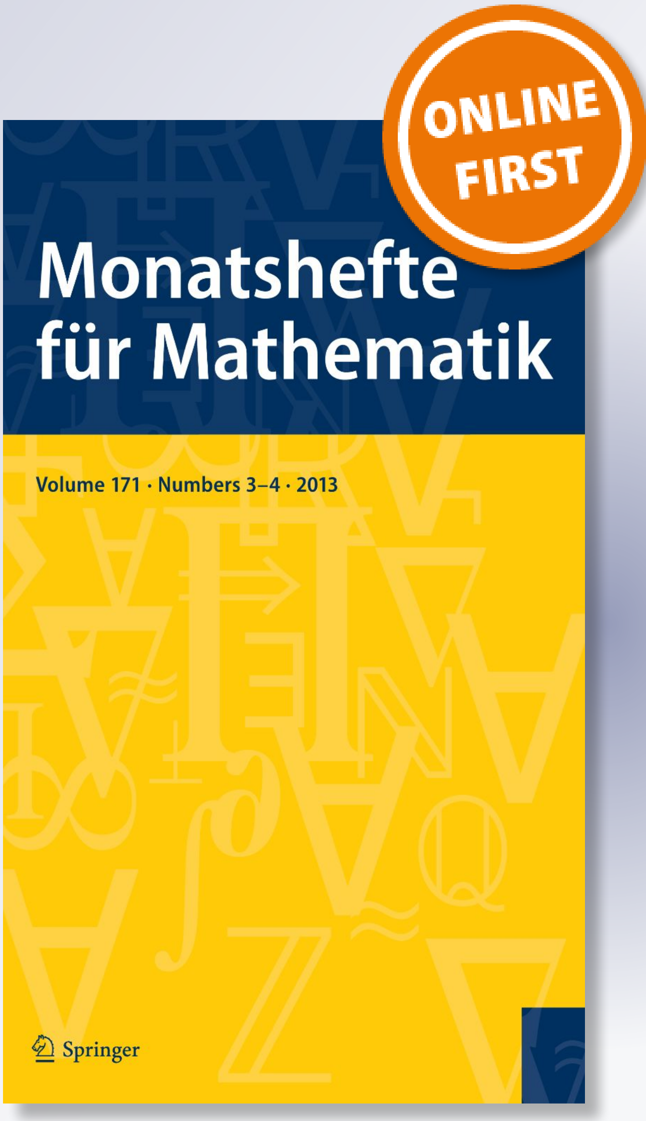

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A note on the product of conjugacy classes of a finite group

Neda Ahanjideh[1]

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Abstract

In Guralnick and Moreto (Conjugacy classes, characters and products of elements, arXiv:1807.03550v1, Theorem 4.2) it has been shown that if $p \neq q$ are two odd primes, $\pi = \{2, p, q\}$ and G is a finite group such that for every π-elements $x, y \in G$ with $(O(x), O(y)) = 1$, $(xy)^G = x^G y^G$, then G does not have any composition factors of order divisible by pq. In this note, inspired by the above result, we show that if p and q are two primes (not necessarily odd) and G is a finite group such that for every p-element x and q-element $y \in G$, $(xy)^G = x^G y^G$, then G does not have any composition factors of order divisible by pq. In particular, we show that if p is an odd prime and G is a finite group such that for every p-element x and 2-element $y \in G$, $(xy)^G = x^G y^G$, then G is p-solvable.

Keywords The product of conjugacy classes · Almost simple groups · Irreducible character degree

Mathematics Subject Classification 20E45 · 20D05 · 20C15

1 Introduction

For a finite group G and $x, y \in G$, let $x^y = y^{-1}xy$ and, x^G and $C_G(x)$ denote the conjugacy class of G containing x and the centralizer of x in G, respectively. The set of irreducible complex characters of G is denoted by $Irr(G)$. Considering the product of conjugacy classes gives us some information about the structure of the group. For instance, in [1, p. 3], Arad and Herzog conjectured that if a finite group G contains a pair (A, B) of conjugacy classes that AB is a conjugacy class too, then G is not simple. This conjecture was proved in [6] in various cases. While as mentioned in [7], simple

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✉ Neda Ahanjideh
ahanjideh.neda@sci.sku.ac.ir

[1] Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran

cannot be replaced by almost simple in the Arad-Herzog conjecture, but we can see that every almost simple group contains some pairs of conjugacy classes which their product is not a conjugacy class (see Lemma 2.5). Dade and Yadav showed that if G is a finite group such that for every $x, y \in G$ with $x^G \neq (y^{-1})^G$, $(xy)^G = x^G y^G$, then G is solvable [3] and they classified such groups. Then Guralnick and Moreto [5] focused on a finite group which the product of every two conjugacy classes of its primary elements with the co-prime orders is a conjugacy class and showed that such groups are solvable. Then they proved that if $p \neq q$ are two odd primes, $\pi = \{2, p, q\}$ and G is a finite group such that for every π-elements $x, y \in G$ with $(O(x), O(y)) = 1$, $(xy)^G = x^G y^G$, then G does not have any composition factors of order divisible by pq. Obviously, this result does not imply that G is either p-solvable or q-solvable. In [5, Paragraph after Theorem 4.2], the authors guessed that in the above result the assumption $2 \in \pi$ can be omitted. In this note, inspired by this impression, we prove the following theorem:

Theorem 1.1 *Let $p \neq q$ be two primes and G be a finite group. If $(xy)^G = x^G y^G$ for every p-element x and q-element $y \in G$, then G does not have any composition factors of order divisible by pq.*

In [5, Theorem 2.5], it has been proved that if p is a prime and G is a finite group such that $x^G y^G = (xy)^G$ for every p-element x and every p'-element of prime power order y, then G is p-solvable. From Theorem 1.1, we can see that:

Corollary 1.2 *Let p be an odd prime and G be a finite group. If $(xy)^G = x^G y^G$ for every p-element x and 2-element $y \in G$, then G is p-solvable.*

Proof Since by Theorem 1.1, G does not have any composition factors of order divisible by $2p$, we get that G is p-solvable. □

Note that in Corollary 1.2, p-solvability cannot be replaced by solvability, for instance, let $G = A \times S$, where A is an abelian p-group and S is a simple p'-group.

2 Main results

Every simple group of Lie type S in characteristic r has an irreducible character of degree $|S|_r$, the order of r-Sylow subgroup of S, which is called the Steinberg character of S.

Lemma 2.1 [10,11] *Let N be a normal subgroup of a group G, and suppose that N is isomorphic to a finite simple group of Lie type. If St is the Steinberg character of N, then St extends to G.*

Lemma 2.2 [5, Lemma 2.3] *Let G be a finite group and let $x, y \in G$. Then $(xy)^G = x^G y^G$ if and only if $\chi(1)\chi(x^a y^b) = \chi(x)\chi(y)$ for every $a, b \in G$ and $\chi \in Irr(G)$.*

Lemma 2.3 [6, Theorem 1.6] *Let G be a finite simple group of Lie type, and let St denote the Steinberg character of G. If $a, b \in G - \{1\}$ are semi-simple elements, then St is not constant on $a^G b^G$.*

Lemma 2.4 [8, Theorem A (Main Theorem)] *Let p and q be distinct odd primes and let G be a finite group. The following statements are equivalent:*

(i) *G contains a composition factor whose order is divisible by pq;*
(ii) *G contains a $(2, p, q)$-triple, where a $(2, p, q)$-triple means a triple (x, y, z) of nontrivial elements in G where x is a 2-element, y a p-element, z an q-element such that $xyz = 1$.*

Lemma 2.5 *Let $p \neq q$ be two primes, $\pi = \{p, q\}$ and S be a non-abelian simple group such that $p, q \mid |S|$. If $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$, then there exist π-elements $x, y \in S$ with $(O(x), O(y)) = 1$ such that $x^G y^G \neq (xy)^G$.*

Proof By way of contradiction, let for every π-elements $x, y \in S$ with $(O(x), O(y)) = 1$, $x^G y^G = (xy)^G$. If S is a Sporadic simple group or Tits group, then the proof follows by checking [2] and [9]. If $S = Alt_n$, $n \geq 5$ and $n \neq 6$, then the proof follows from [4]. Note that $Alt_6 \cong PSL_2(9)$. Now let S be a simple group of Lie type in characteristic r. Thus S has a Steinberg character St which is an irreducible character of S such that $St(u) = |C_S(u)|_r$ for every r'-element $u \in S$ and otherwise, $St(u) = 0$. By Lemma 2.1, St is extendible to G, so there exists $\chi \in Irr(G)$ such that $\chi_S = St$. For every π-elements $x, y \in S$ with $(O(x), O(y)) = 1$, every $a, b \in S$ and every $\varphi \in Irr(G)$, since $x^G y^G = (xy)^G$, Lemma 2.2 forces $\varphi(x)\varphi(y) = \varphi(x^a y^b)\varphi(1)$. Thus $\chi(x)\chi(y) = \chi(1)\chi(x^a y^b)$ and hence, $St(x)St(y) = St(1)St(x^a y^b)$. If $p, q \neq r$, then x and y are semi-simple elements of S, so by Lemma 2.3, St is not constant on $x^S y^S$. On the other hand, for every $a, b \in S$, $St(1)St(x^a y^b) = St(x)St(y)$. This implies that there exists a constant α such that $|S|_r St(x^a y^b) = \alpha$ and hence, St is constant on $x^S y^S$, which is a contradiction.

Therefore, $r \in \{p, q\}$. Without loss of generality, let $r = p$. Thus $St(x)St(y) = 0$ and hence, $St(xy) = 0$. So $p \mid O(xy)$. Now we continue the proof in the following cases:

(i) $p = 2$. Since S is a simple group, there exist at least three prime divisors of the order of S. Thus there exists a prime divisor $t \neq p, q$ of the order of S and hence, Lemma 2.4 shows that there exist a p-element x, a q-element y and a t-element z in $S - \{1\}$ such that $xyz = 1$. Thus $O(xy) = O(z) = t^\alpha$ is not divisible by $p = 2$, a contradiction.
(ii) $p \neq 2$. Then since $2 \mid |S|$, we get from Lemma 2.4 that there exist a p-element x, a q-element y and a 2-element z in $S - \{1\}$ such that $xyz = 1$. Thus $O(xy) = O(z) = 2^\alpha$ is not divisible by p, a contradiction.

These contradictions complete the proof. □

Proof of Theorem 1.1 Let G be a minimal counterexample. Since every quotient of G satisfies the assumption of theorem, we can assume that every minimal normal subgroup of G is a direct product of non-abelian simple groups of order divisible by pq. If $N \cong S_1 \times \cdots \times S_m$ is a minimal normal subgroup of G, then $pq \mid |S_1|$ and by minimality of G, G/N dos not contain any composition factors of order divisible by pq. Hence, N is the unique minimal normal subgroup of G. Now let $T = \{g_1, \ldots, g_k\}$ be a left transversal set of N in G and $M = N_G(S_1)$. Then for every $t \in N$,

$$t^G = \{t^g : g \in G\} = \{t^{g_i n} : 1 \leq i \leq k,\ n \in N\} = \cup_{i=1}^{k} (t^{g_i})^N.$$

Thus for every $a, b \in M$ and π-elements $x, y \in S_1$ with $(O(x), O(y)) = 1$, $x^a y^b \in x^M y^M \subseteq x^G y^G = (xy)^G = \cup_{i=1}^{k}((xy)^{g_i})^N$ and hence, there exist $1 \leq i \leq k$ and $n \in N$ such that $x^a y^b = (xy)^{g_i n}$. Therefore, $(xy)^{g_i} = nx^a y^b n^{-1} \in nS_1 n^{-1} = S_1$, so $xy \in S_1 \cap g_i S_1 g_i^{-1}$. Since $S_1 \trianglelefteq N$, $g_i S_1 g_i^{-1} \trianglelefteq g_i N g_i^{-1} = N$. Therefore, $\{1\} \neq S_1 \cap g_i S_1 g_i^{-1} \trianglelefteq S_1$, so simplicity of S_1 forces $g_i S_1 g_i^{-1} = S_1$ and hence, $g_i \in N_G(S_1) = M$. Note that $N \leq M$. Thus $((xy)^{g_i})^N \subseteq (xy)^M$. This implies that $x^M y^M \subseteq (xy)^M$ and hence, $x^M y^M = (xy)^M$. This guarantees that for every π-elements $x, y \in S_1$ with $(O(x), O(y)) = 1$, $(xy)^M = x^M y^M$ and hence, for every π-elements $\bar{x}, \bar{y} \in S_1$ with $(O(\bar{x}), O(\bar{y})) = 1$, $(\bar{x})^{\bar{M}}(\bar{y})^{\bar{M}} = (\bar{x}\bar{y})^{\bar{M}}$, where $\bar{x}, \bar{y}$ and $\bar{x}\bar{y}$ are the images of $x, y, xy \in S_1$ in $\bar{M} = M/C_G(S_1)$. This allows us to deduce that there exists a group H such that $S_1 \trianglelefteq H \lesssim \mathrm{Aut}(S_1)$ and for every π-elements $x, y \in S_1$, $x^H y^H = (xy)^H$, which is a contradiction with Lemma 2.5. This completes the proof. □

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