China Communications 中国通信

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Journal:	*China Communications*
Manuscript ID	Draft
Manuscript Type:	Original Article
Keywords:	Low-density parity-check code, Edge coloring, Quasi-cyclic LDPC code, Girth, AWGN channel
Speciality:	Low density parity check codes < Channel coding < Information Theory and Coding, Channel coding < Information Theory and Coding

Edge Coloring of Graphs with Applications in Coding Theory

Abstract

In this paper, a new type of edge coloring of graphs together with a greedy algorithm for such an edge coloring is presented to construct some column-weight three low-density parity-check (LDPC) codes whose Tanner graphs are free of 4-cycles. This kind of edge coloring is applied on some well-known classes of graphs such as complete graphs, complete bipartite graphs and disjoint union of complete graphs, to generate some column-weight 3 LDPC codes having flexibility in terms of code length and rate. Interestingly, the constructed $(3, k)$-regular codes with regularities $k = 4, 5, \dots, 22$ have lengthes $n = 12$, 20, 26, 35, 48, 57, 70, 88, 104, 117, 140, 155, 176, 204, 228, 247, 280, 301, 330, having minimum block length compared to the best known similar codes in the literature. Simulation results show that the quasi-cyclic LDPC codes with large girth lifted from the constructed base matrices have good performances over AWGN channel.

Keywords: Low-density parity-check code, Edge coloring, Quasi-cyclic LDPC code, Girth, AWGN channel.

1 Introduction

Low-density parity-check (LDPC) codes [1] are the most promising class of linear codes due to their ease of implementation and excellent performance over noisy channels when decoded with message-passing algorithms [2]. LDPC codes are being considered in numerous applications including digital communication systems [3], MIMO-OFDM systems [4] and magnetic recording channels [5], because of the easy operation and simple decoding which improve the performance. Much research is devoted to characterizing the performance of LDPC codes and designing codes that have good performances. Based on the methods of construction, LDPC codes can be divided into two categories: random codes [7] and structured codes [8]-[23]. Although randomly constructed LDPC codes of large length give excellent bit-error rate (BER) performance [7], the memory required to specify the nonzero elements of such a random matrix can be a major challenge for hardware implementation, while structured LDPC codes can lead to much simpler implementations, particularly for encoding.

To each parity-check matrix H of an LDPC code, the *Tanner graph* $\mathrm{TG}(H)$ [6] is assigned which collects variable nodes and bit nodes associated to the rows and the columns of H, respectively, and each edge connects a variable node to a bit node if the intersection of the corresponding row and column of H has a nonzero entry. We adopt the common notation of referring to a code with the parity-check matrix having column and row weights j and k, respectively, as an (j, k)-regular LDPC code. The design rate of a (j, k)-regular LDPC code is given by $\mathcal{R} = 1 - j/k$.

The *girth* of a code with the given parity-check matrix H, denoted by $g(H)$, is defined as the length of the shortest cycle in $\mathrm{TG}(H)$ and is commonly considered to be one of the most important code parameter which determines the number of independent iterations [1]. Cycles, especially short cycles, in $\mathrm{TG}(H)$ degrade the performance of LDPC decoders, because they affect the independence of the extrinsic information exchanged in the iterative decoding [7]. It is known [6] that if the Tanner graph of the code is free of short cycles, then the iterative sum-product decoding algorithm converges to the optimal solution. In addition, Tanner [6] derives a lower bound on the minimum distance of a code such that this lower bound increases with the girth of the code. Accordingly, the design of LDPC codes with large girth is of great interest and LDPC codes with large girth are to be preferred.

Different approaches have been studied to construct LDPC codes with large girth. Among the random-like approach constructions, the progressive edge growth (PEG) [24] is one of the most successful approaches for the construction of LDPC codes with large girth. PEG algorithm builds a Tanner graph by connecting the graphs nodes edge-by-edge provided the added edge has minimal impact on the girth of the graph. Besides the PEG algorithm and its evolved construction algorithms, quasi-cyclic (QC) LDPC codes are the promising algebraic structured LDPC codes having applications in the storage systems [26], deep-space communications [27], broadband networks [28] and network coding [29]. This is due to their low-complex encoding and efficient parallel iterative decoding, while the existence of short cycles, especially $4-$cycles, is prohibited in their Tanner graph. Some algebraic structured constructions of QC-LDPC codes have been investigated in [8]-[23]. Among the well known structured LDPC codes, finite geometry LDPC codes and LDPC codes constructed from combinatorial designs [14]-[22] are adequate for high-rate LDPC codes. The error correcting performance of these LDPC codes is verified under proper decoding algorithms but they have severe restrictions on flexibly choosing the code rate and length. Also, since finite geometry LDPC codes usually have much redundancy and large weights in their parity-check matrices, they are not suitable for a strictly power-constrained system with iterative message-passing decoding.

In this paper, the concept of the edge coloring of graphs is used to construct some well-structured block designs whose incidence matrices can be considered as the parity-check matrices of some regular and non-regular column-weight three LDPC codes. The family of parity-check matrices so obtained have Tanner graphs which are free of 4-cycles and the class of LDPC codes derived from our construction possesses flexibility in terms of code length and rate, which may have some possible applications to other typical wireless communications such as relaying [26] and MIMO systems [4]. Interestingly, the constructed $(3, k)$-regular codes with regularities $k = 4, 5, \ldots, 22$ have lengthes $n =$ 12, 20, 26, 35, 48, 57, 70, 88, 104, 117, 140, 155, 176, 204, 228, 247, 280, 301, 330, having minimum block length compared to the best known similar codes in the literature. The constructed parity-check matrices can be considered as the base matrices of some QC-LDPC codes with girth at most 18. Simulation results show that the constructed QC-LDPC codes have good performances on AWGN channel and outperform than the random codes [7], PEG LDPC codes [24], codes constructed based on finite fields [9], [12] and also the group ring based QC-LDPC code constructed in [13].

2 Preliminaries and Constructions

To go through the details of the construction, we need some definitions from graph theory. Let $G = (V, E)$ be a graph with vertex and edge sets V and E, respectively. The *degree* of a vertex $v \in V$ is the number of edges incident in v and a graph called k*-regular* if all vertices have degree k. The maximum degree of vertices of a graph G is denoted by $\Delta(G)$, or simply by Δ. Two vertices u and v are adjacent if there is an edge between u and v and we write $\{u, v\} \in E(G)$. For a vertex $v \in V(G)$, we use $N(v)$ to denote the set of all vertices of G which are adjacent to v. An *independent set* in a graph is a set of pairwise nonadjacent vertices. We say that vertex v and edge e are *incident* if v is an endpoint of e. Also, two edges of G are incident if they have a vertex in common. A graph on n vertices in which any two vertices are adjacent is called *complete graph* and denoted by K_n. A graph G is *bipartite* if the set of its vertices can be partitioned into two disjoint sets V_1 and V_2 such that no two vertices within either V_1 or V_2 are connected by an edge. A bipartite graph G with partite sets V_1 and V_2 is denoted by $G = (V_1, V_2)$. A bipartite graph $G = (V_1, V_2)$ is called *complete bipartite graph* if there is an edge between each vertex of V_1 and each vertex of V_2. A complete bipartite graph $G = (V_1, V_2)$ with $|V_1| = n$ and $|V_2| = m$ is denoted by $K_{n,m}$.

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. Path and cycle on n vertices are denoted by P_n and C_n, respectively. The cycle C_n is called even (odd) if n is even (odd).

In graph theory, a k*-edge proper coloring* of an $n-$vertex graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$ is a labeling $f : E(G) \to C$, where $|C| = k$ (often we use $C = \{n + 1, n + 2, \ldots, n + k\}$ and we call it as the *color set*) and incident edges have different colors. The labels are called colors. By the Vizing's theorem [25], the minimum number of colors needed to color the edges of a simple graph is either its maximum degree Δ or $\Delta + 1$ and for bipartite graphs the number of colors is always Δ. Now, consider graph G with a proper edge coloring f. Hereafter, for a vertex v of G we use S_v to denote the set of all colors which appeared on an edge incident to v, i.e. $S_v = \{f(\{v, w\}) : w \in N(v)\}$.

Example 1. (**Complete graphs**) Consider the complete graph K_l with vertex and edge sets $V(K_l) = \{1, 2, \ldots, l\}$ and $E(K_l) = \{\{i, j\} : 1 \leq i < j \leq l\}$. Now, for odd l, color edge $\{i, j\}$ by color $l + ((i + j)(l + 1)/2 - 1 \pmod{l}) + 1$, and for even l, color edge $\{i, j\}$, $1 \leq i < j < l$ by $l + (i + j - 1 \pmod{l - 1}) + 1$ and edge $\{i, l\}$, $1 \leq i < l$, by color $l + (2i - 1 \pmod{l - 1}) + 1$. It is easy to see that this coloring yields a proper edge coloring of K_l with l colors $C = \{l + 1, l + 2, \ldots, 2l\}$ when l is odd, and $l - 1$ colors $C = \{l + 1, l + 2, \ldots, 2l - 1\}$, when l is even. For $l = 4, 5$, proper edge colorings of K_4 and K_5 are shown in Figure 1.

Example 2. (**Complete bipartite graphs**) Consider the complete bipartite graph $K_{l,l}$ with partite sets $V_1 = \{1, 3, \ldots, 2l - 1\}$ and $V_2 = \{2, 4, \ldots, 2l\}$. Coloring each edge $\{2i - 1, 2j\}$, $1 \leq i, j \leq l$, by color $2l + (2(j - i) \pmod{l}) + 1$ yields a proper edge coloring of $K_{l,l}$ with l colors $C = \{2l + 1, 2l + 2, \ldots, 3l\}$. For example, for $l = 3$ such an edge coloring is given in Figure 1.

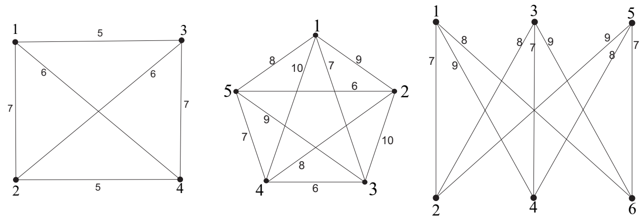


Figure 1: An edge-coloring of K_4, K_5 and $K_{3,3}$.

Example 3. **(Paths and Even Cycles)** Consider the path P_l (resp. even cycle C_l) with vertex set $V = \{1, 2, \ldots, l\}$. Coloring edges of P_l (resp. C_l) alternatively by colors $l + 1$ and $l + 2$ yields a proper edge coloring of path P_l (resp. C_l) with two colors.

Note that in examples 1–3, the set of colors used in the edge colorings are disjoint from the set of vertices of graph. But in general, for an edge-colored graph G with color set C, we may have $V(G) \cap C \neq \emptyset$, i.e. the set of colors can be selected from the vertices of G. In this case, we consider an additional property $\mathcal{P}$ on the edge coloring. We say that the proper edge coloring f of a graph G satisfies *property* $\mathcal{P}$ if for every $u, v \in V(G)$,

$$v \in N(u) \cup S_u \Rightarrow u \notin S_v. \tag{1}$$

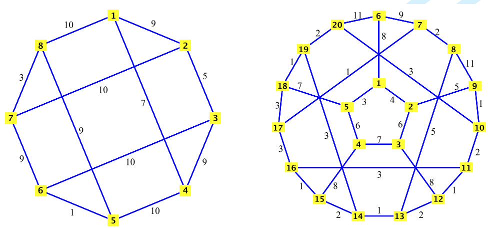
In fact, property (1) says that if v is a neighbor of u or a color appeared on the edges incident with u, then u can not be considered as a color on edges incident with vertex v. For instance, in Figure 2, two edge-colored graphs are presented so that some of colors are selected from the set of vertices and these colorings satisfying property $\mathcal{P}$. In Figure 2, part (a), the set of vertices of G is $V(G) = \{1, \ldots, 8\}$ and the set of colors is $C = \{1, 3, 5, 7, 9, 10\}$. In this figure, $N(1) \cup S_1 = \{2, 4, 7, 8, 9, 10\}$, and so the color 1 can not be appeared on the edges incident with vertices 2,4,7,8.

Hereafter, we suppose that all edge colorings are proper and satisfying property $\mathcal{P}$. Note that if $V(G) \cap C = \emptyset$, then any proper edge coloring of G with color set C satisfies property $\mathcal{P}$. Thus, we consider property $\mathcal{P}$ for all edge colorings in which the color set intersect the vertices of G. In the sequel, an algorithm is presented which greedily provides a proper edge coloring of a simple graph satisfying property $\mathcal{P}$ with the smallest possible number of colors. In fact, Algorithm 1 considers all possible permutations on the edges and for a given specific permutation σ, assigns to each edge the smallest available color satisfying property $\mathcal{P}$, using a new color (a color disjoint from the vertex set) if needed. As considering all possible permutations on the edges of a graph implies a high complexity, we can apply a randomized method to select some permutations such that the number of colors used by the algorithm is small as possible. This algorithm returns the family b_{opt} consisting triples $\{a, b, c\}$ such that $\{a, b\}$ is an edge of the graph and c is the color assigned to this edge.

Example 4. *Let G be the 20-vertex Flower Snark graph, depicted in Figure 2 part (b), with the edge set* $E = \{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 9\}, \{3, 4\}, \{3, 12\}, \{4, 5\}, \{4, 15\}, \{5, 18\}, \{6, 7\}, \{6, 20\}, \{7, 8\}, \{7, 17\}, \{8, 9\}, \{8, 13\}, \{9, 10\}, \{10, 11\}, \{10, 20\}, \{11, 12\}, \{11, 16\},$

Algorithm 1 An edge-coloring of a graph satisfying property $\mathcal{P}$

Require: A simple graph G;
Ensure: A proper edge-coloring of G satisfying property $\mathcal{P}$;
  Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{e_1, \ldots, e_m\}$, where $e_i = \{u_i, v_i\}$, $1 \le u_i, v_i \le n$.
  $A \Leftarrow \{1, \ldots, m + n\}$ and $m_{\max} \Leftarrow n + m$.
  for $\sigma \in S_m$ **do**
    for $i = 1, 2, \ldots, m + n$ and $j = 1, 2, \ldots, m$ **do**
      $s_j \Leftarrow \emptyset$ and $b_i \Leftarrow \{u_{\sigma(i)}, v_{\sigma(i)}\}$;
    end for
    for $i = 1, 2, \ldots, m$ **do**
      set $s_{u_{\sigma(i)}} \Leftarrow s_{u_{\sigma(i)}} \cup \{i\}$ and $s_{v_{\sigma(i)}} \Leftarrow s_{v_{\sigma(i)}} \cup \{i\}$.
    end for
    $B \Leftarrow \emptyset$.
    for $i = 1, \cdots, m$ **do**
      Choose the smallest element $t \in A \setminus (s_{u_{\sigma(i)}} \cup s_{v_{\sigma(i)}})$, and $s_{u_{\sigma(i)}} \Leftarrow s_{u_{\sigma(i)}} \cup \{t\}$, $s_{v_{\sigma(i)}} \Leftarrow s_{v_{\sigma(i)}} \cup \{t\}$;
      $s_t \Leftarrow s_t \cup \{u_{\sigma(i)}, v_{\sigma(i)}\}$, $b_i \Leftarrow b_i \cup \{t\}$ and $B \Leftarrow B \cup b_i$;
    end for
    if the maximum element of B, $\max(B)$, is less than $m_{\max}$ **then**
      $b_{opt} \Leftarrow b$ and $m_{\max} \Leftarrow \max(B)$.
    end if
  end for
  Return b_{opt} as a solution.



(a) (b)

Figure 2: An edge-coloring of the (a) cube graph and (b) Flower-Snark graph, satisfying property $\mathcal{P}$.

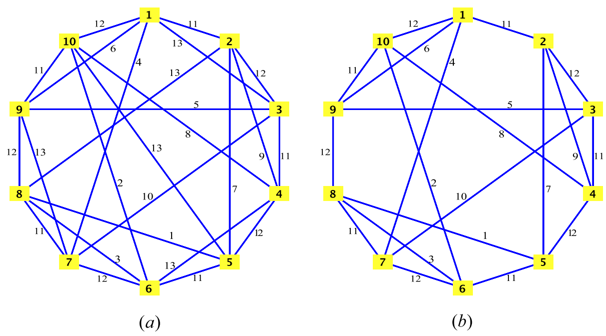


Figure 3: An edge-coloring satisfying property $\mathcal{P}$.

$\{12,13\}$, $\{13,14\}$, $\{14,15\}$, $\{14,19\}$, $\{15,16\}$, $\{16,17\}$, $\{17,18\}$, $\{18,19\}$, $\{19,20\}\}$. *Applying Algorithm 1 on G, we obtain that*
$b_{opt} = \{\{1,2,4\}, \{1,3,5\}, \{1,6,8\}, \{2,3,6\}, \{2,5,9\}, \{3,4,7\}, \{3,8,12\}, \{4,5,6\}, \{4,8,15\}, \{5,7,18\}, \{6,7,9\}, \{6,11,20\}, \{2,7,8\}, \{1,7,17\}, \{8,9,11\}, \{5,8,13\}, \{1,9,10\}, \{2,10,11\}, \{3,10,20\}, \{1,11,12\}, \{3,11,16\}, \{2,12,13\}, \{1,13,14\}, \{2,14,15\}, \{3,14,19\}, \{1,15,16\}, \{2,16,17\}, \{3,17,18\}, \{1,18,19\}, \{2,19,20\}\}$.

In combinatorial mathematics, a $(v,b)-$*design* is a set of *points* $V = \{1,2,\ldots,v\}$ together with a family of size b of k-subsets of V called *blocks*, whose members are chosen to satisfy in some set of properties that are deemed useful for a particular application. The *incidence matrix* of a $(v,b)-$design is a $v \times b$ binary matrix $(h_{i,j})$, in which the rows and columns are corresponding to the points and blocks respectively, such that $h_{i,j} = 1$ if the i-th point belongs to the j-th block and $h_{i,j} = 0$, otherwise. In this paper we are interested in the construction of designs whose incidence matrices are free of 4-cycles i.e. any two points are contained in at most one block.

Let G be an n-vertex graph with m edges and let f be a proper edge coloring of G on the set of colors C satisfying property $\mathcal{P}$ and $|C \setminus V(G)| = t$. Let $\mathcal{B}_f(G)$ denote the set of all triples $\{a, b, f(\{a,b\})\}$, where $e = \{a,b\}$ is an edge of G with color $f(\{a,b\})$. If f is known, we simply denote $\mathcal{B}_f(G)$ by $\mathcal{B}(G)$. It is clear to see that $\mathcal{B}_f(G)$ is a $(n+t, m)-$design. If $\mathcal{H}_f(G)$ denote the $(n+t) \times m$ incident matrix of $\mathcal{B}_f(G)$ then $\mathcal{H}_f(G)$ is the parity-check matrix of a column-weight 3 LDPC code with length m and rate $\mathcal{R} = 1 - \frac{n+t}{m}$. For example, if f, g are the edge colorings of K_4 and $K_{3,3}$, respectively, shown in Figure 1, then $B_f(K_4) = \{\{1,2,7\}, \{3,4,7\}, \{1,3,5\}, \{2,4,5\}, \{1,4,6\}, \{2,3,6\}\}$ and $B_g(K_{3,3}) = \{\{1,2,7\},\{3,4,7\}, \{5,6,7\}, \{2,3,8\}, \{4,5,8\}, \{1,6,8\}, \{1,4,9\}, \{3,6,9\}, \{2,5,9\}\}$, with the following incidence matrices:

$$\mathcal{H}_f(K_4) = \begin{pmatrix} 1&0&1&0&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&0&1\\ 0&1&0&1&1&0\\ 0&0&1&1&0&0\\ 0&0&0&0&1&1\\ 1&1&0&0&0&0 \end{pmatrix}, \quad \mathcal{H}_g(K_{3,3}) = \begin{pmatrix} 1&0&0&0&0&1&1&0&0\\ 1&0&0&1&0&0&0&0&1\\ 0&1&0&1&0&0&0&1&0\\ 0&1&0&0&1&0&1&0&0\\ 0&0&1&0&1&0&0&0&1\\ 0&0&1&0&0&1&0&1&0\\ 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{pmatrix}.$$

Example 5. *Consider graphs G and H shown in Figure 3, parts (a) and (b) with the presented edge colorings f and g, respectively. Clearly*

$B_f(G) = \{\{1, 2, 11\}, \{2, 3, 12\}, \{3, 4, 11\}, \{4, 5, 12\}, \{5, 6, 11\}, \{6, 7, 12\}, \{7, 8, 11\}, \{8, 9, 12\}, \{9, 10, 11\}, \{1, 10, 12\}, \{1, 3, 13\}, \{1, 9, 6\}, \{1, 7, 4\}, \{2, 4, 9\}, \{2, 5, 7\}, \{2, 8, 13\}, \{3, 7, 10\}, \{3, 9, 5\}, \{4, 6, 13\}, \{4, 10, 8\}, \{5, 8, 1\}, \{5, 10, 13\}, \{6, 10, 2\}, \{7, 9, 13\}, \{6, 8, 3\}\}$,

and

$B_g(H) = \{\{1, 2, 11\}, \{2, 3, 12\}, \{3, 4, 11\}, \{4, 5, 12\}, \{5, 6, 11\}, \{6, 7, 12\}, \{7, 8, 11\}, \{8, 9, 12\}, \{9, 10, 11\}, \{1, 10, 12\}, \{1, 9, 6\}, \{1, 7, 4\}, \{2, 4, 9\}, \{2, 5, 7\}, \{3, 7, 10\}, \{3, 9, 5\}, \{4, 10, 8\}, \{5, 8, 1\}, \{6, 10, 2\}, \{6, 8, 3\}\}$,

with the following 13×25 and 12×20 incidence matrices, respectively. Note that $\mathcal{H}_g(H)$ corresponds to the parity-check matrix of a $(3, 5)$-regular code of length 20.

$$\mathcal{H}_f(G) = \begin{pmatrix}
1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&1&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0
\end{pmatrix}, \mathcal{H}_g(H) = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0&0\\
1&1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&1&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1\\
0&0&1&1&0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0\\
0&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&1&1&0&0&0&0&1&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Now, Consider graphs $G_1, G_2, \ldots, G_l$ with edge colorings $f_1, f_2, \ldots, f_l$ receptively, such that for each $i \neq j$, $V(G_i) \cap V(G_j) = \emptyset$. If $\mathcal{G} = (G_1, G_2, \ldots, G_l)$ and $\Gamma = (f_1, f_2, \ldots, f_l)$, then we define $\mathcal{B}_\Gamma(\mathcal{G}) = \bigcup_{i=1}^{l} \mathcal{B}_{f_i}(G_i)$. If Γ is known, we simply denote $\mathcal{B}_\Gamma(\mathcal{G})$ by $\mathcal{B}(\mathcal{G})$. If $\mathcal{H}_\Gamma(\mathcal{G})$ denote the incident matrix of $\mathcal{B}_\Gamma(\mathcal{G})$, then $\mathcal{H}_\Gamma(\mathcal{G})$ is the parity-check matrix of a column-weight 3 LDPC code, denoted by $\mathcal{C}_\Gamma(\mathcal{G})$. In the following, we prove that $\mathcal{C}_\Gamma(\mathcal{G})$ is free of 4-cycles.

Lemma 1. *Let $\mathcal{G} = (G_1, G_2, \ldots, G_l)$ be a class of disjoint graphs with the proper edge colorings $\Gamma = (f_1, f_2, \ldots, f_l)$, respectively. Then $\mathcal{C}_\Gamma(\mathcal{G})$ is free of 4-cycles.*

Proof. Since all graphs $G_1, G_2, \ldots, G_l$ are disjoint, i.e. $V(G_i) \cap V(G_j) = \emptyset$, $\mathcal{C}_\Gamma(\mathcal{G})$ is free of 4-cycles if and only if every $\mathcal{H}_{f_i}(G_i)$, $1 \leq i \leq l$, is free of 4-cycles. Therefore, it is sufficient to prove that for each graph G with the edge coloring f, $\mathcal{H}_f(G)$, is free of 4-cycles. The existence of a 4-cycle in $\mathcal{H}_f(G)$ is equivalent to the existence of two distinct blocks $B_i = \{a_i, b_i, f(\{a_i, b_i\})\}$ and $B_j = \{a_j, b_j, f(\{a_j, b_j\})\}$ such that $|B_i \cap B_j| = 2$, where $\{a_i, b_i\}, \{a_j, b_j\} \in E(G)$ and $f(\{a_i, b_i\})$ is the color of the edge $\{a_i, b_i\}$. Since G is a simple graph and f is a proper edge coloring of G, we may consider the following cases:

Case 1. $a_i = a_j$ and $f(\{a_i, b_i\}) = f(\{a_j, b_j\})$.

In this case, we have incident edges $\{a_i, b_i\}$ and $\{a_i, b_j\}$ having the same color in G. This contradicts the fact that f is a proper edge coloring of G.

Case 2. $a_i = a_j$ and $f(\{a_i, b_i\}) = b_j$.

In this case, $\{a_i, b_i\}$ is colored by b_j and so $b_j \in S_{a_i}$. In addition, $a_i = a_j$ and $\{a_i, b_i\} \in E(G)$ implies that $a_i \in N(b_j)$. Therefore, we obtain that $b_j \in S_{a_i}$ and $a_i \in N(b_j)$, which contradicts property $\mathcal{P}$. This contradiction shows that $\mathcal{C}_\Gamma(\mathcal{G})$ is free of 4-cycles. ■

Proposition 1. *If V is an arbitrary set, then there exist some graph G and edge coloring f of G so that $\mathcal{B}_f(G)$ is a design on V.*

Proof. Let $|V| = n$ and consider $G = P_{n-2}$ with the proper 2-edge coloring f, described in example 3. Label the vertices and colors in G by elements of V. Clearly $\mathcal{B}_f(G)$ define a design on V containing some 3-sets of V. ■

Note that for a given positive integer n, the design used in the proof of Proposition 1 just contains $n-2$ blocks. But, in general we are looking for some graphs G with an edge coloring f such that $\mathcal{B}_f(G)$ contains the maximum possible number of blocks on n points.

Now, let G be an n-vertex graph and let f be an edge coloring of G with color set C'. Set $C = C' \backslash V(G)$. Also let vertices of G be partitioned into disjoint independent sets $V_1, V_2, \ldots, V_l$ such that $|V_i| \geq 3$. By Proposition 1, we may consider $l+1$ graphs $G_1, G_2, \ldots, G_l, G_c$ with edge colorings $f_1, f_2, \ldots, f_l, f_c$, respectively, such that $\mathcal{B}_{f_i}(G_i)$, $1 \leq i \leq l$, is a design on the set V_i and $\mathcal{B}_{f_c}(G_c)$ is a design on the set C. Set $\mathcal{G} = (G_1, G_2, \ldots, G_l, G_c)$ and $\Gamma = (f_1, f_2, \ldots, f_l, f_c)$ and let $\mathcal{B}(G^f, \mathcal{G}^\Gamma) = \mathcal{B}_f(G) \cup \mathcal{B}_\Gamma(\mathcal{G})$. Clearly $\mathcal{B}(G^f, \mathcal{G}^\Gamma)$ is a design on the set $V(G) \cup C$ and if $\mathcal{G} = \emptyset$, then $\mathcal{B}(G^f, \mathcal{G}^\Gamma) = \mathcal{B}_f(G)$. Hereafter, in $\mathcal{B}(G^f, \mathcal{G}^\Gamma)$ we call G as the base graph and $(\mathcal{G}, \Gamma)$ as the generator set. If edge colorings f and Γ are known, then we simply denote $\mathcal{B}(G^f, \mathcal{G}^\Gamma)$ by $\mathcal{B}(G, \mathcal{G})$. It is easy to see that the incidence matrix $\mathcal{H}(G^f, \mathcal{G}^\Gamma)$ of $\mathcal{B}(G^f, \mathcal{G}^\Gamma)$ can be considered as the parity-check matrix of a column-weight three LDPC code with girth 6 which has the following form.

$$\mathcal{H}(G^f, \mathcal{G}^\Gamma) = \begin{array}{c} V_1 \\ \vdots \\ V_l \\ C \end{array} \left(\begin{array}{c|cccc} & \mathcal{H}_{f_1}(G_1) & \cdots & \mathbf{0} & \mathbf{0} \\ & & \ddots & & \\ \mathcal{H}_f(G) & \mathbf{0} & \cdots & \mathcal{H}_{f_l}(G_l) & \mathbf{0} \\ & \mathbf{0} & \cdots & \mathbf{0} & \mathcal{H}_{f_c}(G_c) \end{array} \right).$$

Note: It is worth notice that in the complete graph $G = K_l$ we just consider the incidence matrix of the design on the set of colors i.e. $\mathcal{H}_{f_c}(G_c)$. Thus, if $\mathcal{G} = G_c$ then

$$\mathcal{H}(K_l, G_c) = \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_l \\ C \end{array} \left(\begin{array}{c|c} & 0 \\ & 0 \\ \mathcal{H}_f(G) & \vdots \\ & 0 \\ & \mathcal{H}_{f_c}(G_c) \end{array} \right).$$

For edge colored bipartite graph $G = (V_1, V_2)$, $|V_1|, |V_2| \geq 3$, with color set C, we consider the designs on the partite sets V_1, V_2 and the sets of colors C. Thus, if $\mathcal{G} = (G_1, G_2, G_c)$ then

$$\mathcal{H}(G^f, \mathcal{G}^\Gamma) = \begin{array}{c} V_1 \\ V_2 \\ C \end{array} \left(\begin{array}{c|ccc} & \mathcal{H}_{f_1}(G_1) & \mathbf{0} & \mathbf{0} \\ \mathcal{H}_f(G) & \mathbf{0} & \mathcal{H}_{f_2}(G_2) & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathcal{H}_{f_c}(G_c) \end{array} \right).$$

Example 6. *1. Let $G = K_4$ be the complete graph with $V(K_4) = \{1, 2, 3, 4\}$ and consider the edge coloring of K_4 with color set $C = \{5, 6, 7\}$, shown in Figure 1. Let $G_c = K_2$ with vertex set $\{5, 6\}$ and edge color 7. If $\mathcal{G} = (\emptyset, \emptyset, \emptyset, \emptyset, K_2)$, then $\mathcal{B}(G_c) = \{5, 6, 7\}$ and so $\mathcal{B}(K_4, \mathcal{G}) = \{\{1, 2, 7\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}, \{3, 4, 7\}, \{5, 6, 7\}\}$ with the following incidence matrix $\mathcal{H}(K_4, \mathcal{G})$.*

2. Let $G = K_{3,3}$ with partite sets $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$ and edge-color set $C = \{7, 8, 9\}$. Set $G_1 = K_2$ with vertex set $\{1, 3\}$ colored by 5, $G_2 = K_2$ with vertex set $\{2, 4\}$ and edge of color 6 and $G_3 = K_2$ with vertex set $\{7, 8\}$ and edge of color 9. Thus $\mathcal{B}(G_1) = \{1, 3, 5\}$, $\mathcal{B}(G_2) = \{2, 4, 6\}$ and $\mathcal{B}(G_c) = \{7, 8, 9\}$. If $\mathcal{G} = (K_2, K_2, K_2)$, then $\mathcal{B}(K_{3,3}, \mathcal{G}) = \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}, \{2, 3, 8\}, \{4, 5, 8\}, \{1, 6, 8\}, \{1, 4, 9\}, \{3, 6, 9\}, \{2, 5, 9\}, \{1, 3, 5\}, \{2, 4, 6\}, \{7, 8, 9\}\}$ with the following incidence matrix $\mathcal{H}(K_{3,3}, \mathcal{G})$.

3. Consider $G = C_6$ with vertex set $\{1, 2, \ldots, 6\}$ ordered in clockwise order. Consider partite sets $V_1 = \{1, 3, 5\}$, $V_2 = \{2, 4, 6\}$ and let f be a proper edge coloring of G with color set $C = \{7, 8\}$. Set $G_1 = K_2$ with vertex set $\{1, 3\}$ colored by 5, $G_2 = K_2$ with vertex set $\{2, 4\}$ colored by 6 and $G_c = \emptyset$. If $\mathcal{G} = (K_2, K_2, \emptyset)$, then $\mathcal{B}(C_6, \mathcal{G}) = \{\{1, 2, 7\}, \{2, 3, 8\}, \{3, 4, 7\}, \{4, 5, 8\}, \{5, 6, 7\}, \{6, 1, 8\}, \{1, 3, 5\}, \{2, 4, 6\}\}$ with the following incidence matrix $\mathcal{H}(C_6, \mathcal{G})$.

$$\mathcal{H}(K_4, \mathcal{G}) = \begin{pmatrix} 1110000 \\ 1001100 \\ 0101010 \\ 0010110 \\ 0100101 \\ 0011001 \\ 1000011 \end{pmatrix}, \quad \mathcal{H}(K_{3,3}, \mathcal{G}) = \begin{pmatrix} 100001100100 \\ 100100001010 \\ 010100010100 \\ 010010100010 \\ 001010001100 \\ 001001010010 \\ 111000000001 \\ 000111000001 \\ 000000111001 \end{pmatrix}, \quad \mathcal{H}(C_6, \mathcal{G}) = \begin{pmatrix} 10000110 \\ 11000001 \\ 01100010 \\ 00110001 \\ 00011010 \\ 00001101 \\ 10101000 \\ 01010100 \end{pmatrix}.$$

Example 7. *Let G be the graph on 10 vertices $\{1, 2, \ldots, 10\}$ shown in Figure 3 with the presented edge coloring f on the color set $C = \{1, 2, \ldots, 13\}$. Set $C' = C \setminus V(G) = \{11, 12, 13\}$ and let $G_{c'} = K_2$ with vertex set $\{11, 12\}$ and edge color 13. If $\mathcal{G} = (\emptyset, \ldots, \emptyset, K_2)$, then $\mathcal{B}(G_c) = \{11, 12, 13\}$ and so $\mathcal{B}(G, \mathcal{G}) = \{\{1, 2, 11\}, \{2, 3, 12\}, \{3, 4, 11\}, \{4, 5, 12\}, \{5, 6, 11\}, \{6, 7, 12\}, \{7, 8, 11\}, \{8, 9, 12\}, \{9, 10, 11\}, \{1, 10, 12\}, \{1, 3, 13\}, \{1, 9, 6\}, \{1, 7, 4\}, \{2, 4, 9\}, \{2, 5, 7\}, \{2, 8, 13\}, \{3, 7, 10\}, \{3, 9, 5\}, \{4, 6, 13\}, \{4, 10, 8\}, \{5, 8, 1\}, \{5, 10, 13\}, \{6, 10, 2\}, \{7, 9, 13\}, \{6, 8, 3\}, \{11, 12, 13\}\}$, with the following incidence matrix, which can be considered as the parity-check matrix of a $(3, 6)$-regular code with length 26.*

$$\mathcal{H}(G, \mathcal{G}) = \begin{pmatrix} 11111100000000000000000000 \\ 10000011111100000000000000 \\ 01000010000111100000000000 \\ 00100001000100011100000000 \\ 00010000100010010011000000 \\ 00001000010001001010100000 \\ 00100000100000100000111000 \\ 00010000001001000100010100 \\ 00001001000010000000001110 \\ 00000100010001001010000010 \\ 10000000000100000010010011 \\ 00000110000000010000100101 \\ 01000000001000001001001001 \end{pmatrix}.$$

Now, we can go through the details of the construction. Let G be an n-vertex graph and let f be an edge coloring of G with color set C' and set $C = C' \setminus V(G)$. Also, let vertices of G be partitioned into disjoint independent sets $V_1, V_2, \ldots, V_l$ such that $|V_i| = n_i \geq 3$. For disjoint graphs $G_1, G_2, \ldots, G_l, G_c$ with edge colorings $f_1, f_2, \ldots, f_l, f_c$, receptively, and generator sets $(\mathcal{G}_1, \Gamma_1), \ldots, (\mathcal{G}_l, \Gamma_l), (\mathcal{G}_c, \Gamma_c)$, let $\mathcal{B}(G_i^{f_i}, \mathcal{G}_i^{\Gamma_i})$, $1 \leq i \leq l$, and $\mathcal{B}(G_c^{f_c}, \mathcal{G}_c^{\Gamma_c})$ be constructed such that $\mathcal{B}(G_i^{f_i}, \mathcal{G}_i^{\Gamma_i})$, $1 \leq i \leq l$, is a design on V_i and $\mathcal{B}(G_c^{f_c}, \mathcal{G}_c^{\Gamma_c})$ is a design on C. Let $\mathbf{H}_i$ be the incidence matrix of $\mathcal{B}_i = \mathcal{B}(G_i^{f_i}, \mathcal{G}_i^{\Gamma_i})$, $1 \leq i \leq l$, and $\mathbf{H}_c$ be the incidence matrix of $\mathcal{B}_c = \mathcal{B}(G_c^{f_c}, \mathcal{G}_c^{\Gamma_c})$. Set $\mathbf{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}_c)$ and define:

$$\mathcal{B}(G^f, \mathbf{B}) = \bigcup_{i=1}^{l} \mathcal{B}_i \cup \mathcal{B}_c \cup \mathcal{B}_f(G). \tag{2}$$

The incidence matrix of the constructed design in (2), can be considered as the following form. Since this matrix is based on G and $\mathbf{H} = (\mathbf{H}_1, \ldots, \mathbf{H}_l, \mathbf{H}_c)$, thus we denote this matrix by $\mathcal{H}(G^f, \mathbf{H})$.

$$\mathcal{H}(G^f, \mathbf{H}) = \begin{array}{c} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_l \\ C \end{array} \left(\begin{array}{c|ccccc} & \mathbf{H}_1 & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ & & \cdot & & & \\ \mathcal{H}_f(G) & & & \cdot & & \\ & & & & \cdot & \\ & \mathbf{0} & \mathbf{0} & & \mathbf{H}_l & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{H}_c \end{array} \right).$$

Note: It is worth notice that for edge colored complete graph $G = K_l$, we just consider $\mathbf{H} = \mathbf{H}_c$ and so

$$\mathcal{H}(K_l, \mathbf{H}) = \begin{array}{c} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_l \\ C \end{array} \left(\begin{array}{c|c} & \mathbf{0} \\ & \mathbf{0} \\ & \cdot \\ \mathcal{H}_f(G) & \cdot \\ & \cdot \\ & \mathbf{0} \\ & \mathbf{H}_c \end{array} \right). \tag{3}$$

For edge colored bipartite graph $G = (V_1, V_2)$, we consider $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_c)$ and so

$$\mathcal{H}(G^f, \mathbf{H}) = \begin{array}{c} V_1 \\ V_2 \\ C \end{array} \left(\begin{array}{c|ccc} & \mathbf{H}_1 & \mathbf{0} & \mathbf{0} \\ \mathcal{H}_f(G) & \mathbf{0} & \mathbf{H}_2 & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathbf{H}_c \end{array} \right) \tag{4}$$

If edge coloring f of G is completely known, we simply write $\mathcal{B}(G, \mathbf{B})$ instead of $\mathcal{B}(G^f, \mathbf{B})$. Since $\mathbf{H} = (\mathbf{H}_1, \ldots, \mathbf{H}_l, \mathbf{H}_c)$ and each $\mathbf{H}_\mathbf{i}$ is free of 4-cycles then by a similar argument used in the proof of Lemma 1, we obtain that $\mathcal{H}(G^f, \mathbf{H})$ is free of 4-cycles and so $\mathcal{H}(G^f, \mathbf{H})$ can be considered as the parity-check matrix of a column-weight three LDPC code with girth at least six, denoted by $\mathcal{C}(G^f, \mathbf{H})$. Therefore, with these notations we have the following theorem.

Theorem 1. *For every graph G with edge coloring f, $\mathcal{C}(G^f, \mathbf{H})$ is free of 4-cycles.*

3 Codes Based On the Edge Coloring of Complete Graphs

In this section, we examine the constructed codes by several examples. The constructed codes are based on the edge coloring of complete graph K_l and by (3), it is sufficient to consider $\mathbf{H_c}$, where C is the set of colors in an edge coloring of K_l.

Example 8. *For $i = 7, 8$ let $G = K_i$ be the complete graph with vertex set $\{1, 2, \ldots, i\}$ and edge coloring f introduced in Example 1 with color set $C = \{i+1, i+2, \ldots, i+7\}$. Let $\mathbf{H_c} = \mathcal{H}(K_4, \mathcal{G})$ be the incidence matrix of the design constructed in example 6, part 1 on the points C. Thus*

$$\mathcal{H}(K_7, \mathbf{H_c}) = \left(\begin{array}{c|c} 111111000000000000000 & \\ 100000111110000000000 & \\ 010000100001111000000 & \\ 001000010001000111000 & \mathbf{0} \\ 000100001000100100110 & \\ 000010000100010010101 & \\ 000001000010001001011 & \\ \hline 000000000010010100000 & \\ 010000000000000001100 & \\ 000100010000000000001 & \\ 000001000100100000000 & \mathbf{H_c} \\ 100000000000001010000 & \\ 001000100000000000010 & \\ 000010001001000000000 & \end{array}\right), \quad \mathcal{H}(K_8, \mathbf{H_c}) = \left(\begin{array}{c|c} 1111111000000000000000000000 & \\ 1000000111111000000000000000 & \\ 0100000100000111110000000000 & \\ 0010000010000100001111000000 & \mathbf{0} \\ 0001000001000010001000111000 & \\ 0000100000100001000100100110 & \\ 0000010000010000100010010101 & \\ 0000001000001000010001001011 & \\ \hline 0000010000100010000001000000 & \\ 0000001000010001001000000000 & \\ 1000000000000000100100001000 & \\ 0100000000001000000010100000 & \mathbf{H_c} \\ 0010000100000000000000010010 & \\ 0001000010000000010000000100 & \\ 0000100001000100000000000001 & \end{array}\right).$$

Note that these matrices can be considered as the parity-check matrices of a $(3,6)$ and a $(3,7)$-regular LDPC code with girth 6 having lengths 28 and 35, respectively.

Example 9. *For $i = 9, 10$ let $G = K_i$ be the complete graph with vertex set $\{1, 2, \ldots, i\}$ and edge coloring f introduced in Example 1 with color set $C = \{i+1, i+2, \ldots, i+9\}$. Let $\mathbf{H_c} = \mathcal{H}(K_{3,3}, \mathcal{G})$ be the incidence matrix of the design constructed in example 6 part 2, on the points C. Thus*

$$\mathcal{H}(K_9, \mathbf{H_c}) = \left(\begin{array}{c|c} 111111110000000000000000000000000000 & \\ 100000001111111000000000000000000000 & \\ 010000001000000111111000000000000000 & \\ 001000000100000100000111110000000000 & \\ 000100000010000010000100001111000000 & \mathbf{0} \\ 000010000001000001000010001000111000 & \\ 000001000000100000100001000100100110 & \\ 000000100000010000010000100010010101 & \\ 000000010000001000001000010001001011 & \\ \hline 000000000000001000010001001000000000 & \\ 010000000000000000000000010010100000 & \\ 000100000100000000000000000000001100 & \\ 000001000001000010000000000000000001 & \\ 000000010000010000100010000000000000 & \mathbf{H_c} \\ 100000000000000000001000100100000000 & \\ 001000001000000000000000000001010000 & \\ 000010000010000100000000000000000010 & \\ 000000100000100001000100000000000000 & \end{array}\right),$$

$$\mathcal{H}(K_{10}, \mathbf{H_c}) = \left(\begin{array}{c|c} 111111111000000000000000000000000000000000000 & \\ 100000000111111110000000000000000000000000000 & \\ 010000000100000001111111000000000000000000000 & \\ 001000000010000001000000111111000000000000000 & \\ 000100000001000000100000100000111110000000000 & \mathbf{0} \\ 000010000000100000010000010000100001111000000 & \\ 000001000000010000001000001000010001000111000 & \\ 000000100000001000000100000100001000100100110 & \\ 000000010000000100000010000010000100010010101 & \\ 000000001000000010000001000001000010001001011 & \\ \hline 000000010000001000001000010000000010000000000 & \\ 000000001000000100000100001000100000000000000 & \\ 100000000000000000000010000100010000001000000 & \\ 010000000000000010000000000010001001000000000 & \\ 001000000100000000000000000000000100100001000 & \mathbf{H_c} \\ 000100000010000000000001000000000000010100000 & \\ 000010000001000001000000000000000000000010010 & \\ 000001000000100000100000000001000000000000100 & \\ 000000100000010000010000100000000000000000001 & \end{array}\right),$$

which can be considered as the parity-check matrices of a $(3, 8)$-regular and a $(3, 9)$-regular LDPC code with girth 6 having lengths 48 and 57, respectively.

Example 10. *Let $G = K_{13}$ with vertex set $\{1, 2, \ldots, 13\}$ and edge coloring f introduced in Example 1 with color set $C = \{14, 15, \ldots, 26\}$. Let $\mathbf{H_c}$ be the incidence matrix of the design constructed in example 7 on the points $C = \{14, \ldots, 26\}$. Thus $\mathcal{H}(K_{13}, \mathbf{H_c})$ can be considered as the parity-check matrix of a $(3, 12)$-regular LDPC code with girth 6 and length 104.*

$$\mathcal{H}(K_{13}, \mathbf{H_c}) = \left(\begin{array}{c|c} \mathcal{H}_f(K_{13}) & \begin{array}{c} \mathbf{0}_{13\times 26} \\ \\ \mathbf{H_c} \end{array} \end{array} \right).$$

Example 11. *Let $G = K_{15}$ (resp. $G = K_{16}$) with vertex set $\{1, 2, \ldots, 15\}$ (resp. $\{1, 2, \ldots, 16\}$) and edge coloring f introduced in Example 1 with color set $C = \{16, \ldots, 30\}$ (resp. $C = \{17, \ldots, 31\}$). Let $\mathbf{H_c}$ be the incidence matrix of the design constructed in example 8 based on K_8 on the points C. Thus for $n = 15, 16$ we have*

$$\mathcal{H}(K_n, \mathbf{H_c}) = \left(\begin{array}{c|c} \mathcal{H}_f(K_n) & \begin{array}{c} \mathbf{0}_{n\times 35} \\ \\ \mathbf{H_c} \end{array} \end{array} \right).$$

Therefore, $\mathcal{H}(K_{15}, \mathbf{H_c})$ and $\mathcal{H}(K_{16}, \mathbf{H_c})$ are the parity-check matrices of a girth-6 regular column-weight three LDPC code with regularities $14, 15$, respectively.

Example 12. *Let $G = K_{19}$ with vertex set $V_1 = \{1, 2, \ldots, 19\}$ and edge coloring f introduced in Example 1 with color set $C = \{20, 21, \ldots, 38\}$. Let $\mathbf{H_c}$ be the incidence matrix of the design constructed in example 9 based on K_{10}. Therefore, $\mathcal{H}(K_{19}, \mathbf{H_c})$ is the incidence matrix of a regular column-weight three LDPC code with regularity 18 and girth 6.*

4 Codes Based On the Edge Coloring of Complete Bipartite Graphs

In this section, we examine the constructed codes based on the edge coloring of complete bipartite graph $K_{n,n}$ and by (4), it is sufficient to consider $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_c$, where C is the set of colors in an edge coloring of $K_{n,n}$.

Example 13. *Let $G = K_{6,6}$ with vertex set $V_1 = \{1, 2, \ldots, 6\}$ and $V_2 = \{7, 8, \ldots, 12\}$ and edge coloring f introduced in Example 1 with color set $C = \{13, 11, \ldots, 18\}$. Set $\mathbf{H_1} = \mathbf{H_2} = \mathbf{H_c}$ be the incidence matrix of the design $\mathcal{B}_f(K_{2,2})$, where f is the edge coloring introduced in example 2. Thus if $\mathbf{H} = (\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_c})$, then $\mathcal{H}(K_{6,6}, \mathbf{H})$ can be considered as the parity-check matrix of a $(3, 8)$-regular LDPC code with girth 6 having length 48.*

$$
\mathcal{H}(K_{6,6},\mathbf{H})=\left(\begin{array}{c|c|c|c}
111111000000000000000000000000000000 & 1001 & 0000 & 0000 \\
100000100000100000100000100000100000 & 1100 & 0000 & 0000 \\
000000111111000000000000000000000000 & 0110 & 0000 & 0000 \\
010000010000010000010000010000010000 & 0011 & 0000 & 0000 \\
000000000000111111000000000000000000 & 1010 & 0000 & 0000 \\
001000001000001000001000001000001000 & 0101 & 0000 & 0000 \\ \cline{2-4}
000000000000000000111111000000000000 & 0000 & 1001 & 0000 \\
000100000100000100000100000100000100 & 0000 & 1100 & 0000 \\
000000000000000000000000111111000000 & 0000 & 0110 & 0000 \\
000010000010000010000010000010000010 & 0000 & 0011 & 0000 \\
000000000000000000000000000000111111 & 0000 & 1010 & 0000 \\
000001000001000001000001000001000001 & 0000 & 0101 & 0000 \\ \cline{2-4}
100000010000001000000100000010000001 & 0000 & 0000 & 1001 \\
010000001000000100000010000000110000 & 0000 & 0000 & 1100 \\
001000000100000010000001100000010000 & 0000 & 0000 & 0110 \\
000100000010000001100000010000001000 & 0000 & 0000 & 0011 \\
000010000000110000010000001000000100 & 0000 & 0000 & 1010 \\
000001100000010000001000000100000010 & 0000 & 0000 & 0101
\end{array}\right)
$$

Example 14. *Let $G = K_{7,7}$ with vertex set $V_1 = \{1, 2, \ldots, 7\}$ and $V_2 = \{8, 9, \ldots, 14\}$ and edge coloring f introduced in Example 2 with color set $C = \{15, 16, \ldots, 21\}$. Set $\mathbf{H_1} = \mathbf{H_2} = \mathbf{H_c}$ be the incidence matrix of the design constructed in example 6. Thus if $\mathbf{H} = (\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_c})$, then $\mathcal{H}(K_{7,7}, \mathbf{H})$ can be considered as the parity-check matrix of a $(3, 10)$-regular LDPC code with girth 6 having length 70.*

$$
\mathcal{H}(K_{7,7},\mathbf{H})=\left(\begin{array}{c|c|c|c}
 & 1110000 & 0000000 & 0000000 \\
 & 1001100 & 0000000 & 0000000 \\
 & 0101010 & 0000000 & 0000000 \\
 & 0010110 & 0000000 & 0000000 \\
 & 0100101 & 0000000 & 0000000 \\
 & 0011001 & 0000000 & 0000000 \\
 & 1000011 & 0000000 & 0000000 \\ \cline{2-4}
 & 0000000 & 1110000 & 0000000 \\
 & 0000000 & 1001100 & 0000000 \\
 & 0000000 & 0101010 & 0000000 \\
\mathcal{H}_f(K_{7,7}) & 0000000 & 0010110 & 0000000 \\
 & 0000000 & 0100101 & 0000000 \\
 & 0000000 & 0011001 & 0000000 \\
 & 0000000 & 1000011 & 0000000 \\ \cline{2-4}
 & 0000000 & 0000000 & 1110000 \\
 & 0000000 & 0000000 & 1001100 \\
 & 0000000 & 0000000 & 0101010 \\
 & 0000000 & 0000000 & 0010110 \\
 & 0000000 & 0000000 & 0100101 \\
 & 0000000 & 0000000 & 0011001 \\
 & 0000000 & 0000000 & 1000011
\end{array}\right)
$$

Note that using this matrix as $\mathbf{H_c}$ in $\mathcal{H}(K_{21}, \mathbf{H_c})$ and $\mathcal{H}(K_{22}, \mathbf{H_c})$ presented in (3), one can construct a $(3, 20)$-regular and a $(3, 21)$-regular LDPC code with girth 6 having lengths 280 and 301, respectively.

Example 15. *Let $G = K_{8,8}$ with vertex set $V_1 = \{1, 2, \ldots, 8\}$ and $V_2 = \{8, 9, \ldots, 16\}$ and edge coloring f introduced in Example 2 with color set $C = \{16, 17, \ldots, 24\}$. Let $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}_c$ be the incidence matrix of the design constructed in example 6 part 3, and $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_c)$. Therefore, $\mathcal{H}(K_{8,8}, \mathbf{H})$ is the incidence matrix of a regular column-weight three LDPC code with regularity 10 and girth 6.*

Example 16. *Let $G = K_{9,9}$ with vertex set $V_1 = \{1, 2, \ldots, 9\}$ and $V_2 = \{10, 11, \ldots, 18\}$ and edge coloring f introduced in Example 2 with color set $C = \{21, 22, \ldots, 30\}$. Let $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}_c$ be the incidence matrix of the design constructed in example 6 part 2, and $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_c)$. Therefore, $\mathcal{H}(K_{9,9}, \mathbf{H})$ is the parity-check matrix of a regular column-weight three LDPC code with regularity 13 and girth 6.*

Example 17. *Let $G = K_{12,12}$ with vertex set $V_1 = \{1, 2, \ldots, 12\}$ and $V_2 = \{13, 14, \ldots, 24\}$ and edge coloring f introduced in Example 2 with color set $C = \{25, 26, \ldots, 36\}$. Let $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}_c$ be the incidence matrix of $B_g(H)$ constructed in example 5 and $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_c)$.*

Therefore, $\mathcal{H}(K_{12,12}, \mathbf{H})$ *is the parity-check matrix of a regular column-weight three LDPC code with regularity* 17*, length 204 and girth 6.*

5 Codes Based On the Edge Coloring of Disjoint Complete Graphs

For odd l, let N_l be the graph with vertex set $V(N_l) = \{1, 2, , \dots, 3l\}$ containing three disjoint complete graphs $K_l^{(1)}$, $K_l^{(2)}$, $K_l^{(3)}$ with vertex sets $V(K_l^{(1)}) = \{1, 2, \dots, l\}$, $V(K_l^{(2)}) = \{l + 1, l + 2, \dots, 2l\}$ and $V(K_l^{(3)}) = \{2l + 1, 2l + 2, \dots, 3l\}$. Since l is odd, any edge coloring of a copy $K_l^{(k)}$, $k = 1, 2, 3$, need l colors. Use $V(K_l^{(2)})$ as the color set $K_l^{(1)}$, use $V(K_l^{(3)})$ as the color set $K_l^{(2)}$ and finally use $V(K_l^{(1)})$ as the color set $K_l^{(3)}$. For this purpose, it is sufficient to color each edge $\{i, j\}$ in $K_l^{(1)}$ by color $l + (\frac{(i+j)(l+1)}{2} - 1 \pmod{l}) + 1$, each edge $\{i, j\}$ in $K_l^{(2)}$ by color $2l + (\frac{(i+j)(l+1)}{2} - 1 \pmod{l}) + 1$ and finally each edge $\{i, j\}$ in $K_l^{(3)}$, by color $(\frac{(i+j)(l+1)}{2} - 1 \pmod{l}) + 1$. Also, one can easily see that the set of vertices $V_i = \{i, l + i, 2l + i\}$, $1 \le i \le l$, form an independent set in N_l and so the vertices of N_l can be partitioned into independent sets of size 3. Now for each i, $1 \le i \le l$, let $G_i = K_2$ with vertex set $\{i, l + i\}$ which is colored by color $2l + i$. If $\mathcal{G} = (G_1, G_2, \dots, G_l)$, then $\mathcal{H}(G_l, \mathcal{G})$ is the parity-check matrix of a 4-cycle free regular LDPC code with regularity $(3l - 1)/2$. If the rows of $\mathcal{H}(N_l, \mathcal{G})$ are indexed by the vertices in the sets $V_1, V_2, \dots, V_l$, then $\mathcal{H}(N_l, \mathcal{G})$ has the following form in which, $\mathbf{1}$ and $\mathbf{0}$ are 3×1 full one and full zero matrices, respectively. Since $\mathcal{G} = (G_1, G_2, \dots, G_l)$ is known, so $\mathcal{H}(N_l, \mathcal{G})$ just depends on N_l and thus we use $\overline{\mathcal{H}}(N_l)$ instead of $\mathcal{H}(N_l, \mathcal{G})$.

$$\overline{\mathcal{H}} = \mathcal{H}(N_l, \mathcal{G}) = \begin{array}{c} V_1 \\ V_2 \\ \vdots \\ V_l \end{array} \left(\begin{array}{c|cccc} & \mathbf{1} & \mathbf{0} & \mathbf{0} \cdots & \mathbf{0} \\ & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathcal{H}_f(N_l) & & & \ddots & \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array} \right)$$

Example 18. *If* $l = 3$*, then* $V(N_3) = \{1, 2, \dots, 9\}$ *and sets* $V_1 = \{1, 4, 7\}$*,* $V_2 = \{3, 6, 9\}$ *and* $V_3 = \{2, 5, 8\}$ *are independent sets in* N_3*. The corresponding design based on* N_3 *is* $\{\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 9\}$, $\{4, 6, 8\}$, $\{5, 6, 7\}$, $\{3, 7, 8\}$, $\{2, 7, 9\}$, $\{1, 8, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}\}$ *with the following parity-check matrix* $\overline{\mathcal{H}}(N_3)$*.*

Example 19. *If* $l = 5$*, then* $V(N_5) = \{1, 2, \dots, 15\}$ *and sets* $V_1 = \{1, 6, 11\}$*,* $V_2 = \{2, 7, 12\}$*,* $V_3 = \{3, 8, 13\}$*,* $V_4 = \{4, 9, 14\}$*,* $V_5 = \{5, 10, 15\}$ *are independent sets in* N_5*. The corresponding design based on* N_5 *is* $\{\{1, 2, 9\}$, $\{6, 7, 14\}$, $\{4, 11, 12\}$, $\{1, 3, 7\}$, $\{6, 8, 12\}$, $\{2, 11, 13\}$, $\{1, 4, 10\}$, $\{6, 9, 15\}$, $\{5, 11, 14\}$, $\{1, 5, 8\}$, $\{6, 10, 13\}$, $\{3, 11, 15\}$, $\{2, 3, 10\}$, $\{7, 8, 15\}$, $\{5, 12, 13\}$, $\{2, 4, 8\}$, $\{7, 9, 13\}$, $\{3, 12, 14\}$, $\{2, 5, 6\}$, $\{7, 10, 11\}$, $\{1, 12, 15\}$, $\{3, 4, 6\}$, $\{8, 9, 11\}$, $\{1, 13, 14\}$, $\{3, 5, 9\}$, $\{8, 10, 14\}$, $\{4, 13, 15\}$, $\{4, 5, 7\}$, $\{9, 10, 12\}$, $\{2, 14, 15\}$, $\{1, 6, 11\}$, $\{2, 7, 12\}$, $\{3, 8, 13\}$, $\{4, 9, 14\}$, $\{5, 10, 15\}\}$*, with the following parity-check matrix* $\overline{\mathcal{H}}(N_5)$*.*

$$\overline{\mathcal{H}}(N_3) = \begin{array}{c} 1\\4\\7\\3\\6\\9\\2\\5\\8 \end{array} \begin{pmatrix} 1\,1\,0\,0\,0\,0\,0\,0\,1\,1\,0\,0\\ 0\,0\,1\,1\,1\,0\,0\,0\,0\,1\,0\,0\\ 0\,0\,0\,0\,0\,1\,1\,1\,0\,1\,0\,0\\ 1\,0\,1\,0\,0\,0\,0\,1\,0\,0\,1\,0\\ 0\,1\,0\,1\,0\,1\,0\,0\,0\,0\,1\,0\\ 0\,0\,0\,0\,1\,0\,1\,0\,1\,0\,1\,0\\ 0\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0\,1\\ 1\,0\,0\,0\,1\,1\,0\,0\,0\,0\,0\,1\\ 0\,0\,0\,1\,0\,0\,0\,1\,1\,0\,0\,1 \end{pmatrix},$$

$$\overline{\mathcal{H}}(N_5) = \begin{array}{c} 1\\6\\11\\2\\7\\12\\3\\8\\13\\4\\9\\14\\5\\10\\15 \end{array} \begin{pmatrix} 11110000000000000000000001100010000\\ 00000011001111000000000000000010000\\ 00000000000000001101111000000010000\\ 10001110000000000000100000001001000\\ 01000000011000111000000000000001000\\ 00000000000100000001000111000101000\\ 01001001100000000000010100000000100\\ 00010100000100100110000000000000100\\ 00000000000001010000100100110000100\\ 00100101010000000000100000001000010\\ 10000000100010010100000000000100010\\ 00000000001000000010010010101000010\\ 00010010110000000000010100000000001\\ 00101000000000000010010100000100001\\ 00000000000010100000100101100000001 \end{pmatrix}.$$

In general, we have the following theorem.

Theorem 2. *For any odd l, the incidence matrix $\overline{\mathcal{H}}(N_l)$ can be considered as the parity-check matrix of a 4-cycle free regular LDPC code with regularity $(3l-1)/2$.*

Table 1 provides some $(3,k)$-regular LDPC codes, $1 \leq k \leq 22$, constructed from some edge colored graphs. The constructed $(3,k)$-regular codes with regularities $k = 4, 5, \ldots, 22$ have lengthes $n = 12, 20, 26, 35, 48, 57, 70, 88, 104, 117, 140, 155, 176, 204, 228, 247, 280, 301, 330$, having minimum block length compared to the best known similar codes in the literature.

6 QC-LDPC CODES WITH GIRTHS At Most 18

Although LDPC codes have good error performances over AWGN channel, their encoding complexity was a drawback for their implementation until the recent two decades and the invention of the quasi-cyclic LDPC (QC-LDPC) codes. QC-LDPC codes have advantages over other types of LDPC codes in hardware implementation of encoding [18] and decoding [9], [15]. These features have made the design of QC-LDPC codes an attractive research area and lots of methods, including algebraic methods, are proposed for constructing QC-LDPC codes. The parity-check matrix of the constructed column-weight 3 LDPC codes are free of 4-cycles and so they can be considered as the mother matrix of some column-weight three QC-LDPC code with maximum achievable girth 18 [8]. In the sequel, we give an algorithm to generate such QC-LDPC codes with maximum girth 18.

Let N and s be nonnegative integers with $0 \leq s \leq N-1$. By an $N \times N$ *circulant permutation matrix (CPM) shifted by* s, $\mathcal{I}_N^s$, we mean the matrix obtained from $N \times N$ identity matrix $\mathcal{I}_N$ by shifting each row s positions to the bottom, that is $\mathcal{I}_N^0 = \mathcal{I}_N$ and for $1 \leq s \leq N-1$,

$$\mathcal{I}_N^s = \begin{pmatrix} 0 & \mathcal{I}_s \\ \mathcal{I}_{N-s} & 0 \end{pmatrix}.$$

For simplicity, $\mathcal{I}_N^s$ will be denoted by $\mathcal{I}^s$, when N is known.

k	N	R	Base Graph	Parity-check matrix
4	12	0.25	$K_{3,3}$	example 6, part 2
5	20	0.40	Figure 3 part (b)	example 7
6	26	0.50	Figure 3 part (a)	example 7
7	35	0.57	K_8	example 8
8	48	0.62	K_9	example 9
9	57	0.67	K_{10}	example 9
10	70	0.70	$K_{7,7}$	example 14
11	88	0.73	$K_{8,8}$	example 15
12	104	0.74	K_{13}	example 10
13	117	0.77	$K_{9,9}$	example 16
14	140	0.78	K_{15}	example 11
15	155	0.80	K_{16}	example 11
16	176	0.81	N_{11}	example 2
17	204	0.82	$K_{12,12}$	example 17
18	228	0.83	K_{19}	example 12
19	247	0.84	N_{13}	example 2
20	280	0.85	K_{21}	example 14
21	301	0.86	K_{22}	example 14
22	330	0.863	N_{15}	example 2

Table 1: $(3, k)$-regular codes with regularity k, length N an rate R

For a given integer N and the base matrix H with c nonzero elements, we define a (c, N)-*slope vector* as a vector S of length c such that each component of S belongs to $\mathbb{Z}_N = \{0, 1, \cdots, N-1\}$. Now, let N be a positive integer and $H = (h_{i,j})_{p\times q}$ be the binary base matrix with c nonzero elements $h_{i_1,j_1} = h_{i_2,j_2} = \ldots = h_{i_c,j_c} = 1$. For given (c, N)-slope vector $S = (s_1, s_2, \ldots, s_c)$, let $\mathcal{H}_{N,H,S}$ be the $pN \times qN$ matrix obtained from the base matrix H by replacing the zero elements with the zero $N \times N$ matrix and the non-zero element h_{i_k,j_k} by $\mathcal{I}^{s_k}$.

Let H be a $n \times m$ binary base matrix with c nonzero elements and $G = \mathrm{TG}(H)$ be the corresponding Tanner graph with the edge set $E = \{e_1, \ldots, e_c\}$. For sufficiently large positive inter N, we give a deterministic algorithm generating $(c, N)-$slope vector $S = (s_1, \ldots, s_c)$ inductively (each s_i corresponds to the edge e_i) such that the girth of $\mathrm{TG}(\mathcal{H}_{N,H,S})$ is at least $2g$, $g \geq 3$. Set $A_1 = \{0, 1, \ldots, N-1\}$ and choose $s_1 \in A_1$ arbitrary. Now, let $A_1, \ldots, A_{k-1}$ be chosen and $(s_1, \ldots, s_{k-1}) \in A_1 \times \cdots \times A_{k-1}$. At step k, we choose A_k containing all elements $s_k \in \{0, \ldots, N-1\}$ such that $g(s_1, \ldots, s_k) \geq 2g$, where $g(s_1, \ldots, s_k)$ is the minimum of $2l$, such that for some $2l-$length closed walk $W = e_{i_1}e_{i_2}\ldots e_{i_{2l}}$, $e_{i_{2l+1}} = e_{i_1}$, $k \in \{i_1, i_2, \ldots, i_{2l}\} \subseteq \{1, \ldots, k\}$, $i_j \neq i_{j+1}$, $1 \leq j \leq 2l$, we have $\sum_{j=1}^{2l}(-1)^j s_{i_j} = 0 \pmod{N}$. The algorithm is summarized as follows.

Algorithm II. Generating (c, N)-slope vectors

1. Let N and c be positive integers. Set $A = \{0, \ldots, N-1\}$, $k = 1$, $A_1 = A$.
2. If $k = 0$ then $N \to N + 1$ and go to step 1.
3. If $A_k = \emptyset$, let $A_{k-1} \to A_{k-1} - \{s_{k-1}\}$, $k \to k-1$, and go to step 2.
4. Choose an arbitrary element $s_k \in A_k$.
5. If $k < c$, then set $A_{k+1} = \{s \in A : g(s_1, \ldots, s_k, s) \geq 2g\}$, $k \to k+1$ and go to step 3.
6. Print $S = (s_1, \ldots, s_c)$ as a solution.

For a column-weight three LDPC code with base matrix $H = (h_{i,j})_{v\times b}$, a $(c, N)-$slope vector $S = (s_1, \ldots, s_c)$, $c = 3b$, can be generated from Algorithm II column-by-column starting from the first column traversing each column from top to bottom. For this purpose, let in the $j-$th column of H, $h_{i_1,j} = h_{i_2,j} = h_{i_3,j} = 1$, $i_1 < i_2 < i_3$, be the non-zero elements of this column with the corresponding slopes $s_{j_{i_1}}$, $s_{j_{i_2}}$ and $s_{j_{i_3}}$, respectively. Without loss of generality, we may suppose that $s_{j_{i_1}} = 0$, otherwise we can subtract $s_{j_{i_1}}$ from $s_{j_{i_2}}$ and $s_{j_{i_2}}$ to obtain new slopes 0, $s_{j_{i_2}} - s_{j_{i_1}}$, and $s_{j_{i_3}} - s_{j_{i_1}}$. Therefore, hereafter, we may suppose that $s_i = 0$, for $i \equiv 1 \pmod 3$ and so a $(c, N)-$slope vector $S = (s_1, \ldots, s_c)$, $c = 3b$, can be denoted by a length $2b-$vector $\mathcal{S} = (s_2, s_3, s_5, s_6, \ldots, s_{c-1}, s_c)$. Applying Algorithm II on the constructed base matrices generated in Examples 4, 6 part 2, 7 and 8, Table 2 provides some slope vectors S corresponding to some QC-LDPC codes with girth at most 18.

Graph H	$v \times b$	regularity	$\mathcal{R}$	N	Girth	Slope vector S
Figure 2 (b) (Example 4)	20×30	-	0.33	1871	16	[0, 0, 0, 0, 0, 0, 1, 3, 7, 0, 10, 0, 14, 0, 16, 43, 38, 0, 42, 0, 60, 82, 0, 0, 107, 232, 170, 0, 201, 55, 315, 0, 437, 0, 37, 409, 139, 147, 577, 553, 737, 0, 743, 270, 629, 0, 57, 212, 275, 0, 698, 332, 483, 609, 805, 465, 731, 388, 781, 1055]
$K_{3,3}$ (Example 6)	9×12	4	0.25	3162	18	[0, 0, 0, 0, 0, 0, 1, 0, 2, 5, 9, 22, 35, 0, 38, 62, 109, 225, 228, 516, 563, 1189, 829, 1890]
Figure 3 (a) (Example 7)	13×26	6	0.5	1441	14	[0, 0, 0, 0, 0, 1, 0, 2, 0, 4, 0, 7, 0, 0, 0, 8, 0, 11, 1, 17, 4, 0, 21, 32, 45, 60, 23, 64, 32, 105, 58, 10, 117, 146, 108, 223, 144, 187, 78, 261, 138, 348, 279, 389, 262, 564, 346, 782, 456, 883, 255, 1001]
K_7 (Example 8)	14×28	6	0.5	1221	14	[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 7, 12, 10, 6, 11, 19, 28, 0, 14, 29, 40, 70, 56, 31, 80, 96, 51, 105, 99, 167, 157, 175, 145, 53, 178, 214, 199, 249, 69, 152, 170, 375, 237, 399, 220, 483, 352, 655, 392, 747, 140, 604]

Table 2: List of slope vectors S to generate some QC-LDPC codes with rate $\mathcal{R}$ and CPM-size N.

7 Simulation results

Using software available online [30], we have obtained simulation results on additive white Gaussian noise (AWGN) channel with BPSK modulation and the decoding algorithm is the sum-product algorithm. This section provides some BER comparisons between the error performance of the constructed QC-LDPC codes on the one hand and random-like counterparts [7], PEG LDPC codes [24], codes constructed based on finite fields [9], [12] and the error performance of a group ring based QC-LDPC code constructed in [13], on the other hand.

In the figures, by $Rand(k \times n)$ we mean the 4-cycle free Mackay random code of length n and dimension k and by $PEG(n, \mathtt{tg}b)$ we mean the length n code generated by the progressive edge growth algorithm having target girth b. In addition, $C(m, G, \mathtt{g}b)$ is used to denote the constructed girth-b QC-LDPC code based on graph G having CPM-size m.

Applying Algorithm II on the base matrices constructed in Example 6, part 2, and Example 8, some QC-LDPC codes with same lengths and rates but different girths are constructed and the BER performance comparisons of these codes (with 20 iterations) are presented in Fig. 4. The figure confirms the superiority of the constructed codes with large girth and also confirms that the girth has a direct impact on the codes performance. As shown in Fig. 4, the constructed codes with large girths have good performances over AWGN channel.

In Figu. 5, the parity-check matrix of the code in Example 9 (based on K_9) is used as the base matrix of a girth-8 QC-LDPC code generated by Algorithm II. The performance of this code (with 30 iterations) is shown in Fig. 5. At the BER of 10^{-6}, the code performs 1.8 dB

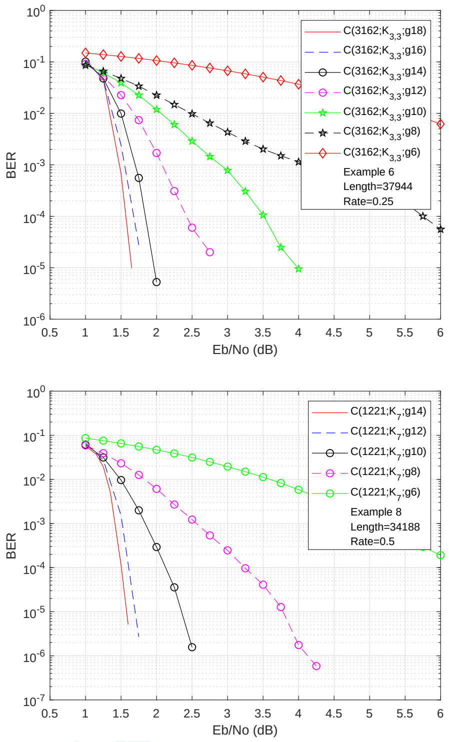


Figure 4: The BER performance comparisons between the QC-LDPC codes with different girths having base matrices constructed in Example 6, part 2, and Example 8.

from the Shannon limit and it outperforms the random code $Rand(774 \times 2064)$, the PEG code $PEG(2064; tg8)$ and the $(2040, 1279)$-LDPC code constructed in [13] based on Z_8, a group of order 8.

Consider the girth-8 $(3, 10)$-regular QC-LDPC code of length 2030 lifted from the base matrix constructed in Example 14. The BER performance of this code is presented in Fig. 6 with 30 iterations. Included in this figure, the 4-cycle free $(2032, 1439)$-LDPC code constructed in [9] (Example 1) based on finite fields. As the figure shows, there is a close competition between the constructed code and the random code $Rand(609 \times 2030)$, the PEG code $PEG(2030; tg8)$ and the code in [9].

Fig. 7 provides a comparison between the girth-8 QC-LDPC code with the base matrix constructed in Example 5, on the one hand and the random code $Rand(726 \times 3872)$, the PEG code $PEG(3872; tg8)$ and the 4-cycle free $(3969, 3243)$-LDPC code with minimum distance at least 63 constructed in [12] (Example 3) based on finite fields, on the other hand. The performance of this code (with 50 iterations) is shown in Fig. 7. In this figure, all compared codes have length 3872 and rate 0.817 and at the BER of 10^{-6}, our code performs 1.3 dB from the Shannon limit, while the $(3969, 3243)$-LDPC code constructed in [12] has 1.6 dB far away the Shannon limit. Morever, we see that the constructed code slightly outperforms its corresponding random code.

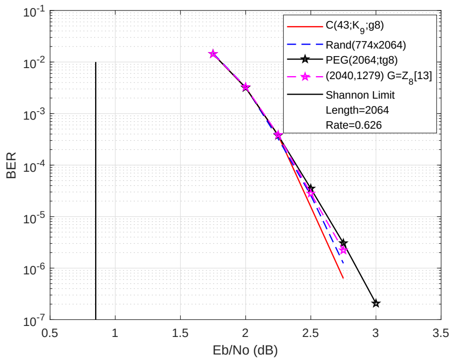


Figure 5: The BER performance comparisons between the constructed QC-LDPC code with base matrix in Example 9 against random, PEG and the $(2040, 1279)$-LDPC code constructed in [13].

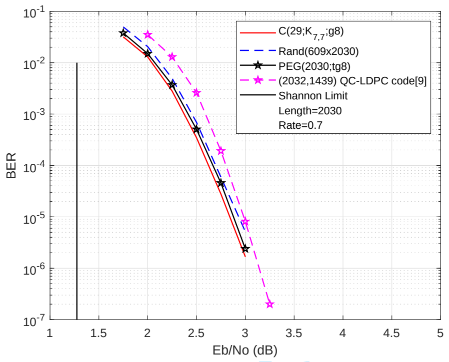


Figure 6: The BER performance comparisons between the constructed QC-LDPC code with base matrix in Example 14 against random, PEG and the 4-cycle free $(2032, 1439)$-LDPC code constructed in [9].

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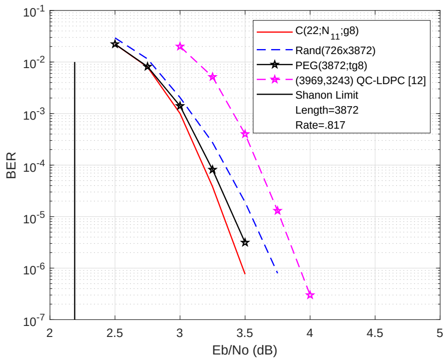

Figure 7: The BER performance comparisons between the constructed QC-LDPC code with base matrix in Section 5 against random, PEG and the $(3969, 3243)$-LDPC code constructed in [12].

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