Thompson's conjecture on conjugacy class sizes for the simple group $PSU_n(q)$

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Abstract

We show that if G is a finite centerless group with the same conjugacy class sizes as $PSU_n(q)$, then $G \cong PSU_n(q)$ and so verify a conjecture attributed to John G. Thompson.

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1 Introduction

Let $cs(G)$ denote the set of conjugacy class sizes of a finite group G.

In 1988, John G. Thompson posed the following conjecture which appears as Problem 12.38 of [10].

Conjecture. *If S is a finite simple group and G is a finite group such that $Z(G) = 1$ and $cs(G) = cs(S)$, then G is isomorphic to S.*

In [1, 2, 3, 4, 5, 6, 8, 9, 15], it has been shown that the conjecture is true for many finite simple groups. We prove the following.

Main Theorem. *If G is a finite group such that $Z(G) = 1$ and $cs(G) = cs(PSU_n(q))$, then $G \cong PSU_n(q)$.*

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2 Definitions and preliminary results

Let H be a finite group. For $x \in H$, $cl_H(x)$ and $C_H(x)$ denote the conjugacy class in H containing x and the centralizer of x in H, respectively. Also, let $\pi(H)$ and $\omega(H)$ be the set of prime divisors of $|H|$ and the set of orders of elements of H, respectively. For $r \in \pi(H)$ (resp. $\pi \subseteq \pi(H)$), $O_r(H)$ (resp. $O_\pi(H)$) is the largest normal r-subgroup (resp. π-subgroup) of H and $O_{r'}(H)$ is the largest normal r'-subgroup of H. Also, $\mathrm{Syl}_r(H)$ denotes the set of r-Sylow subgroups of H.

For a prime r and a natural number a, $|a|_r$ is the r-part of a, i.e., $|a|_r = r^t$, if $r^t \| a$, $|a|_{r'} = a/|a|_r$ is the r'-part of a. If π is a set of primes, then put $|a|_\pi = \prod_{r\in\pi} |a|_r$ and $|a|_{\pi'} = a/|a|_\pi$. Define $\mathrm{sgn}(-1) = -$ and $\mathrm{sgn}(+1) = +$. Sometimes, we use $GL_n^+(q)$ and $GL_n^-(q)$ for $GL_n(q)$ and $GU_n(q)$, respectively.

Throughout this paper, let p be a prime, $q = p^k$, $n \geq 3$ be a natural number such that $(n, q) \neq (3, 2)$ and let G be a finite group such that $Z(G) = 1$ and $cs(G) = cs(PSU_n(q))$. All other notations are borrowed from [7] and [12].

Definition 2.1 *For an integer m with $|m| > 1$ and an odd prime r such that $\gcd(m, r) = 1$, $\exp_r(m)$ denotes the multiplicative order of m modulo r, that is the smallest natural number i with $m^i \equiv 1 \pmod r$. For an odd integer m, we put $\exp_2(m) = 1$ if $m \equiv 1 \pmod 4$ and $\exp_2(m) = 2$, otherwise. A prime r with $\exp_r(m) = i$ is a primitive prime divisor of $m^i - 1$. Let $Z_i(m)$ be the set of all primitive prime divisors of $m^i - 1$.*

Lemma 2.2 (Zsigmondy Theorem) [21, 16] *Let m be an integer with $|m| > 1$. For every positive integer i, there is a primitive prime divisor of $m^i - 1$, except for the pairs $(m, i) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}$.*

Lemma 2.3 *Let r, s, t and u be distinct prime divisors of the order of the finite group H, $K = O_{\{r,s\}}(H)$ and $K_s \in \mathrm{Syl}_s(K)$.*

(i) *If x is a non-trivial s-element of K such that $x \in K_s$, then $|cl_H(x)|_{r'} < |K|_s$.*

(ii) *If $\bar{M} = M/K$ is a normal t-subgroup of $\bar{H} = H/K$, then there exist $M_t \in \mathrm{Syl}_t(M)$ and a non-trivial u-element $y \in H$ such that $M_t \leq N_H(K_s)$ and $y \in N_H(K_sM_t)$.*

Proof. Let $K_r \in \mathrm{Syl}_r(K)$. Then $K = K_rK_s$ and so, by Frattini's argument, $H = KN_H(K_s) = K_rN_H(K_s)$ and hence $[H : N_H(K_s)]$ is an r-number. Since $x \in K_s \trianglelefteq N_H(K_s)$, $cl_{N_H(K_s)}(x) \subset K_s$. Thus $\frac{|cl_H(x)|}{|cl_H(x)|_r} \leq \frac{|cl_H(x)|[C_H(x):C_{N_H(K_s)}(x)]}{[H:N_H(K_s)]} = |cl_{N_H(K_s)}(x)| < |K_s|$. Therefore, $|cl_H(x)|_{r'} < |K|_s$, as required in (i). Now we prove (ii). Since $H = K_rN_H(K_s)$ and $u \in \pi(H) - \{r\}$, $u \mid |N_H(K_s)|$. Also, $K_r \leq M$ and hence, the Dedekind modular law shows that $M = M \cap H = M \cap (K_rN_H(K_s)) = K_r(M \cap N_H(K_s))$. Therefore, there exists $M_t \in \mathrm{Syl}_t(M)$ such that $M_t \leq N_H(K_s)$ and hence, $K_sM_t \leq H$. On the other hand, $M = M_tK \trianglelefteq H$ and hence, the Dedekind modular law shows that

$$M_tN_K(K_s) = M_t(K \cap N_H(K_s)) = (M_tK) \cap N_H(K_s) = M \cap N_H(K_s) \trianglelefteq N_H(K_s).$$

Thus Frattini's argument gives that

$$N_H(K_s) = N_{N_H(K_s)}(M_t)M_tN_K(K_s) = N_{N_H(K_s)}(M_t)N_K(K_s).$$

Since K is a $\{r, s\}$-group and $u \mid |N_H(K_s)|$, we deduce that $u \mid |N_{N_H(K_s)}(M_t)|$ and hence, $N_{N_H(K_s)}(M_t) = N_H(K_s) \cap N_H(M_t)$ contains a non-trivial u-element y. Consequently, $y \in N_H(K_sM_t)$, as claimed in (ii). $\Box$

In the following lemma, we collect some known facts used frequently.

Lemma 2.4 *Let H be a finite group, N a normal subgroup of H and $x, y \in H$.*

(i) *If $xy = yx$ and $\gcd(O(x), O(y)) = 1$, then $C_H(xy) = C_H(x) \cap C_H(y)$. In particular, $C_H(xy) \leq C_H(x)$ and $|cl_H(x)|$ divides $|cl_H(xy)|$;*

(ii) *if $|C_H(x) \cap N| = 1$, then $|N|$ divides $|cl_H(x)|$;*

(iii) *if $x \in N$, then $|cl_N(x)|$ divides $|cl_H(x)|$;*

(iv) *if $\gcd(|N|, O(x)) = 1$, then $C_{H/N}(xN) = C_H(x)N/N$;*

(v) *if $r||H/N|$, $r \nmid |N|$ (r is a prime and $r \neq p$), $p^e|||N|$ and $p^t|||C_N(R)|$, where $R \in \mathrm{Syl}_r(H)$, then $r|p^{e-t} - 1$;*

(vi) *if N is the π-group, for some $\pi \subseteq \pi(H)$, and x is the π'-element of H of a prime power order, then $|cl_H(x)|_{\pi'}$ divides $|cl_{H/N}(xN)|$.*

Proof. (i)-(iii) are straightforward and we obtain (iv) from [11, Theorem 1.6.2]. For the proof of (v), let $P \in \mathrm{Syl}_p(N)$. Since by Frattini's argument, $H = N_H(P)N$, we can assume that $R \in N_H(P)$. Let $Q \in \mathrm{Syl}_p(C_N(R))$ such that $Q \leq P$. Therefore, $|P| \equiv |Q| \pmod r$, so $r \mid p^{e-t} - 1$, as required in (v). For the proof of (vi), applying (iv) shows that $C_{H/N}(xN) = C_H(x)N/N$ and hence $|cl_{H/N}(xN)| = [H/N : C_H(x)N/N] = \frac{|H||C_N(x)|}{|C_H(x)||N|} = \frac{|cl_H(x)|}{[N:C_N(x)]}$ is divisible by $|cl_H(x)|_{\pi'}$, as desired. $\Box$

Lemma 2.5 [2, Lemma 2.7(i)] *Let $r \in Z_n(q)$ and let x be a non-central element of $GL_n(q)$ such that $r \mid |C_{GL_n(q)}(x)|$. If m is the smallest natural number with $O(x) \mid q^m - 1$, then $C_{GL_n(q)}(x) \cong GL_{n/m}(q^m)$.*

In the following lemmas, $GF(q)$ is the field with q elements, $\mathrm{diag}(a_1, ..., a_m)$ is a diagonal matrix with numbers $a_1, a_2, \ldots, a_m$ on a diagonal, $\mathrm{bd}(A_1, A_2, \ldots, A_m)$ denotes a block-diagonal matrix with square blocks $A_1, A_2, \ldots, A_m$ and C^t denotes the transpose of a square matrix C.

Lemma 2.6 *Let t be a natural number such that $2t \mid n$ and let $B \in GL_t(q^2)$ such that $O(B) \mid q^{2t} - 1$ and for every $1 \leq l < 2t$, $O(B) \nmid q^l - (-1)^l$. If $C = \mathrm{bd}(B, \ldots, B) \in GL_{n/2}(q^2)$ and τ is a field automorphism of $GL_{n/2}(q^2)$, then C^τ and $(C^t)^{-1}$ are not conjugate in $GL_{n/2}(q^2)$.*

Proof. Let $\overline{GF}(q^2)$ be the algebraic closure of the field of order q^2 and let ξ be an element of $GF(q^{2t})$ of order $O(B)$. There is $g \in GL_t(\overline{GF}(q^2))$ such that $B = g^{-1}\mathrm{diag}(\xi, \xi^{q^2}, \ldots, \xi^{q^{2(t-1)}})g \in GL_t(q^2)$ (see [17, Lemma 5]). Thus there exists $g_1 \in GL_{n/2}(\overline{GF}(q^2))$ such that

$$C = g_1^{-1}\mathrm{diag}(\xi, \xi^{q^2}, \ldots, \xi^{q^{2(t-1)}}, \ldots, \xi, \xi^{q^2}, \ldots, \xi^{q^{2(t-1)}})g_1.$$

If C^τ and $(C^t)^{-1}$ are conjugate in $GL_{n/2}(q^2)$, then we can assume that there exists $h = (h_{ij}) \in GL_{n/2}(\overline{GF}(q^2))$ such that

$$\begin{aligned} h^{-1} \quad \mathrm{diag} \quad &(\xi^q, \xi^{q^3}, \ldots, \xi^{q^{2(t-1)+1}}, \ldots, \xi^q, \xi^{q^3}, \ldots, \xi^{q^{2(t-1)+1}})h = \\ \mathrm{diag} \quad &(\xi^{-1}, \xi^{-q^2}, \ldots, \xi^{-q^{2(t-1)}}, \ldots, \xi^{-1}, \xi^{-q^2}, \ldots, \xi^{-q^{2(t-1)}}). \end{aligned} \tag{1}$$

Since $\det(h) \neq 0$, there exists $1 \leq j \leq n$ such that $h_{1j} \neq 0$. On the other hand, (1) forces $\xi^q h_{1j} = h_{1j}\xi^{-q^{2l}}$, where $0 \leq l \leq t-1$ and $l \equiv j - 1 \pmod t$. Therefore, $\xi^q = \xi^{-q^{2l}}$ and hence $(\xi^q)^{q^{2l-1}+1} = 1$. Thus $O(\xi^q) = O(\xi) = O(B) \mid q^{2l-1} + 1 = q^{2l-1} - (-1)^{2l-1}$. But

$2l-1 \leq 2(t-1)-1 < 2t$, which is a contradiction by our assumption on $O(B)$. This shows that C^{τ} and $(C^t)^{-1}$ are not conjugate in $GL_{n/2}(q^2)$. $\Box$

Lemma 2.7 *Let $r \in Z_n(-q)$. If x is an element of $GU_n(q)$ of order r, then $C_{GU_n(q)}(x)$ is a cyclic group of order $q^n-(-1)^n$.*

Proof. We prove this lemma in two cases.

Case I. Let $n=2t$. It is easy to check that $r \in Z_t(q^2)$. Let C be an element of $GL_t(q^2)$ of order r. Since $|GL_t(q^2)|_r = |GU_n(q)|_r$ and $GU_n(q) = \{T \in GL_n(q^2) : T^{\tau_1} J_n T^t = J_n\}$, where τ_1 is a field automorphism of $GL_n(q^2)$ of order 2, I_t is an identity matrix in $GL_t(q^2)$ and $J_n = \begin{pmatrix} \mathbf{0} & I_t \\ I_t & \mathbf{0} \end{pmatrix}$, the second Sylow theorem allows us to assume that $x = \mathrm{bd}(C^{\tau}, (C^t)^{-1})$, where τ is a field automorphism of $GL_t(q^2)$ of order 2. By Lemma 2.6, C^{τ} and $(C^t)^{-1}$ are not conjugate in $GL_t(q^2)$ and hence, $C_{GL_n(q^2)}(x) = \{\mathrm{bd}(h_1,h_2) : (h_1)^{\tau}, (h_2^t)^{-1} \in C_{GL_t(q^2)}(C)\}$. Thus $C_{GU_n(q)}(x) = \{\mathrm{bd}(h_1^{\tau}, (h_1^t)^{-1}) : h_1 \in C_{GL_t(q^2)}(C)\} \cong C_{GL_t(q^2)}(C)$. So Lemma 2.5 shows that $C_{GU_n(q)}(x) \cong GL_1(q^n)$, which is a cyclic group of order $q^n-1 = q^n-(-1)^n$.

Case II. Let n be odd. Then $r \in Z_n(q^2)$. Since $GU_n(q) = \{T \in GL_n(q^2) : T^{\tau_1} T^t = I_n\}$, where τ_1 is a field automorphism of $GL_n(q^2)$ of order 2, $x \in GL_n(q^2)$. Lemma 2.5 shows that $C_{GL_n(q^2)}(x) \cong GL_1(q^{2n})$. Note that $GL_1(q^{2n}) = GF(q^{2n}) - \{0\}$. Thus τ_1 can be considered as an involutory field automorphism of $GF(q^{2n})$. Therefore, $C_{GU_n(q)}(x) \cong \{h \in GF(q^{2n})-\{0\} : h^{\tau_2} h^t = 1\} = GU_1(q^n)$, where τ_2 is an involutory field automorphism of $GL_1(q^{2n})$ induced by τ_1.

Therefore, $C_{GU_n(q)}(x)$ is a cyclic group of order $q^n-(-1)^n$, as desired. $\Box$

Lemma 2.8 *Let $r \in Z_n(-q)$. If x is a non-central element of $GU_n(q)$, then either $r \nmid |C_{GU_n(q)}(x)|$ or there exists a divisor m of n such that $C_{GU_n(q)}(x) \cong GL^{\epsilon}_{n/m}(q^m)$, where $m \neq 1$ and $\epsilon = \mathrm{sgn}((-1)^m)$. In the latter case, if $(n,q)=(4,2)$, then $m=4$.*

Proof. Let $r \mid |C_{GU_n(q)}(x)|$. Then $C_{GU_n(q)}(x)$ contains an element y of the order r. Therefore, $x \in C_{GU_n(q)}(y)$. By Lemma 2.7, $C_{GU_n(q)}(y)$ is a cyclic group of the order $q^n-(-1)^n$. Let $C_{GU_n(q)}(y)$ be generated by α. Since $x \in C_{GU_n(q)}(y)$, we deduce that $O(x)$ divides $q^n-(-1)^n$. Let m be the smallest natural number such that $O(x)$ divides $q^m-(-1)^m$. Then m divides n, by [19, Lemma 6(iii)].

Case I. Let $m = 2t$ be even. It is known that $GL_t(q^2)$ contains an element, say B, of order $O(x)$. Set $C := \mathrm{bd}(B, \ldots, B) \in GL_{n/2}(q^2)$ and $A := \mathrm{bd}(C^\tau, (C^t)^{-1})$, where τ is a field automorphism of $GL_{n/2}(q^2)$ of the order 2. Lemma 2.6 shows that C^τ and $(C^t)^{-1}$ are not conjugate in $GL_{n/2}(q^2)$ and hence, $C_{GL_n(q^2)}(A) = \{\mathrm{bd}(h_1, h_2) : (h_1)^\tau, (h_2^t)^{-1} \in C_{GL_{n/2}(q^2)}(C)\}$. On the other hand, we can assume that $GU_n(q) = \{T \in GL_n(q^2) : T^{\tau_1} J_n T^t = J_n\}$, where τ_1 is a field automorphism of $GL_n(q^2)$ of order 2, $I_{n/2}$ is an identity matrix in $GL_{n/2}(q^2)$ and $J_n = \begin{pmatrix} \mathbf{0} & I_{n/2} \\ I_{n/2} & \mathbf{0} \end{pmatrix}$. Therefore, $A \in GU_n(q)$ and $C_{GU_n(q)}(A) = \{\mathrm{bd}(h_1^\tau, (h_1^t)^{-1}) : h_1 \in C_{GL_{n/2}(q^2)}(C)\} \cong C_{GL_{n/2}(q^2)}(C)$. Since $r \in Z_{n/2}(q^2)$, Lemma 2.5 shows that $C_{GU_n(q)}(A) \cong GL_{n/2t}(q^{2t}) = GL_{n/m}(q^m)$.

Case II. Let m be odd. It is known that $GU_m(q)$ contains an element, namely B, of the order $O(x)$. By our assumption on $O(x)$, we see that B is an irreducible element of $GL_m(q^2)$ and since $GU_m(q) = \{T \in GL_m(q^2) : T^\tau T^t = I_m\}$, where τ is a field automorphism of $GL_m(q^2)$ of the order 2, we have $B^\tau B^t = I_m$. Set $A = \mathrm{bd}\underbrace{(B, \ldots, B)}_{n/m-\text{times}} \in GL_n(q^2)$. For the field automorphism τ_1 of $GL_n(q^2)$ of the order 2, $A^{\tau_1} A^t = I_n$ and hence, $A \in GU_n(q)$. Since B is an irreducible element of $GL_m(q^2)$, Schur's lemma guarantees that $C_{GL_n(q^2)}(A) = \{h = (h_{ij}) : h_{ij} \in C_{GL_m(q^2)}(B) \cup \{0\}$, for every $1 \leq i, j \leq n/m\}$. Again by the irreducibility of B, we get that $C_{GL_m(q^2)}(B) \cup \{0\}$ is isomorphic to $GF(q^{2m})$. Thus τ can be considered as an involutory field automorphism of $GF(q^{2m})$. Therefore, $C_{GU_n(q)}(A) = \{h = (h_{ij}) : h_{ij} \in C_{GL_m(q^2)}(B) \cup \{0\}$, for every $1 \leq i, j \leq n/m$ and $h^{\tau_2} h^t = I_{n/m}\} \cong GU_{n/m}(q^m)$, where τ_2 is an involutory field automorphism of $GL_{n/m}(q^{2m})$ induced by τ.

On the other hand, $GL^\epsilon_{n/m}(q^m)$ contains an element of the order $q^n - (-1)^n$ and hence we may assume that $y \in C_{GU_n(q)}(A)$. Thus both $A, x \in C_{GU_n(q)}(y) = \langle \alpha \rangle$. Since $O(A) = O(x)$ and $\langle \alpha \rangle$ contains exactly one subgroup of a given order, we have $\langle A \rangle = \langle x \rangle$ and hence, $C_{GU_n(q)}(x) = C_{GU_n(q)}(A) \cong GL^\epsilon_{n/m}(q^m)$, as desired.

If $(n, q) = (4, 2)$, then $r = 5$. If $r \mid |C_{GU_n(q)}(x)|$, then $C_{GU_n(q)}(x)$ contains a non-trivial r-element y. So $|C_{GU_n(q)}(y)| = 15$ and $|Z(GU_n(q))| = 3$. Thus x is a product of a central element and a non-trivial r-element. This shows that $|C_{GU_n(q)}(x)| = |C_{GU_n(q)}(y)| = 15$, as claimed. $\Box$

Corollary 2.9 *Let $r \in Z_n(-q)$. If x is a non-central element of $SU_n(q)$, then either $r \nmid |C_{SU_n(q)}(x)|$ or there exists a divisor $m \neq 1$ of n such that $|C_{SU_n(q)}(x)| = |GL^\epsilon_{n/m}(q^m)|/(q+1)$,* *where* $\epsilon = \mathrm{sgn}((-1)^m)$. *In the latter case, if* $(n, q) = (4, 2)$, *then* $m = 4$.

Proof. It follows immediately from Lemma 2.8 and the fact that if α is an element of the order $q^n - (-1)^n$ of $GU_n(q)$, then $[\langle\alpha\rangle : \langle\alpha\rangle \cap SU_n(q)] = [GU_n(q) : SU_n(q)] = q + 1$. □

Corollary 2.10 *Let* $r \in Z_n(-q)$. *If* x *is a non-trivial element of* G, *then either* $|cl_G(x)|_r = |PSU_n(q)|_r$ *or there exists a divisor* $m \neq 1$ *of* n *such that* $|cl_G(x)| = \frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|}$, *where* $\epsilon = \mathrm{sgn}((-1)^m)$ *and either* $\beta = 1$ *or* $\gcd(O(x), \gcd(m, q + 1)) \neq 1$ *and* $\beta \mid \gcd(q + 1, m)$. *In the latter case, if* $(n, q) = (4, 2)$, *then* $m \neq 2$.

Proof. It follows immediately from Corollary 2.9. □

Lemma 2.11 *Let* $n > 2$. *If* $r \in Z_{n-1}(-q)$, *then for every non-trivial* $x \in G$, *either* $|cl_G(x)|_r = |PSU_n(q)|_r$ *or there exists a divisor* m *of* $n-1$ *such that* $|cl_G(x)| = \frac{|GU_n(q)|}{(q+1)|GL^{\epsilon}_{(n-1)/m}(q^m)|}$, *where* $\epsilon = \mathrm{sgn}((-1)^m)$. *Also, if* $q + 1 \mid n$, *then* $m \neq 1$.

Proof. The same argument used in the proof of Lemma 2.8 completes the proof. □

Lemma 2.12 [6, Lemma 2.9] *Let* H *be a finite centerless group with* $r \in \pi(H)$ *and let* $\alpha \in cs(H)$ *be maximal in* $cs(H)$ *by divisibility.*

(i) *If for every* $\beta \in cs(H)$, $|H|_r > |\beta|_r$, *then there exists a non-trivial* r*-element* $u \in H$ *such that* $|cl_H(u)|$ *divides* α.

(ii) *If* $\mathrm{Max}\{|\beta|_r : \beta \in cs(H)\} = r^t$ *and for every* $\beta \in cs(H) - \{1\}$ *with* $|\beta|_r < r^t$, *we have* $|\beta|_{r'} \nmid \alpha$, *then* $|H|_r = r^t$.

Lemma 2.13 (i) $\frac{|GU_n(q)|}{(q^n-(-1)^n)}, \frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})} \in cs(G)$. *Moreover,*

$$\frac{|GU_n(q)|}{(q^n - (-1)^n)}, \frac{|GU_n(q)|}{(q + 1)(q^{n-1} - (-1)^{n-1})}$$

are maximal in $cs(G)$ *by divisibility;*

(ii) *if* $t \in \pi(G)$ *such that* $|G|_t > |PSU_n(q)|_t$, *then there exist* t*-elements* $x_n, x_{n-1} \in G$ *such that* $|cl_G(x_n)|$ *divides* $\frac{|GU_n(q)|}{(q^n-(-1)^n)}$ *and* $|cl_G(x_{n-1})|$ *divides* $\frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})}$;

(iii) $|PSU_n(q)|$ *divides* $|G|$ *and* $\pi(PSU_n(q)) = \pi(G)$.

Proof. From Corollary 2.10 and Lemma 2.11, $\frac{|GU_n(q)|}{(q^n-(-1)^n)}, \frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})} \in cs(G)$. Now suppose by contradiction that $\frac{|GU_n(q)|}{(q^n-(-1)^n)}$ is not maximal in $cs(G)$ by divisibility. Since $cs(G) = cs(PSU_n(q))$, we conclude that there exists $x \in PSU_n(q)$ such that $|cl_{PSU_n(q)}(x)| \neq \frac{|GU_n(q)|}{(q^n-(-1)^n)}$ and $\frac{|GU_n(q)|}{(q^n-(-1)^n)}$ divides $|cl_{PSU_n(q)}(x)|$. Thus $|C_{PSU_n(q)}(x)|$ divides $\frac{(q^n-(-1)^n)}{\gcd(n,q+1)(q+1)}$, so x is a semi-simple element of $PSU_n(q)$. Thus there exists a maximal torus T of $PSU_n(q)$ containing x and hence, $T \leq C_{PSU_n(q)}(x)$. Therefore, $|T|$ divides $\frac{(q^n-(-1)^n)}{\gcd(n,q+1)(q+1)}$ and hence, considering the orders of maximal tori of $PSU_n(q)$ (see [18]) shows that $|T| = \frac{(q^n-(-1)^n)}{\gcd(n,q+1)(q+1)}$. Therefore, $|C_{PSU_n(q)}(x)| = \frac{(q^n-(-1)^n)}{\gcd(n,q+1)(q+1)}$, which is a contradiction to our assumption. The same reasoning can be applied to prove that $\frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})}$ is maximal in $cs(G)$ by divisibility, as wanted in (i). Now (ii) follows from (i) and Lemma 2.12(i). Finally, by (i), $\mathrm{lcm}(\frac{|GU_n(q)|}{(q^n-(-1)^n)}, \frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})}) = |PSU_n(q)|$ divides $|G|$ and applying the same argument given in the proof of [3, Corollary 2.8] shows that $\pi(G) \subseteq \pi(PSU_n(q))$, hence $\pi(PSU_n(q)) = \pi(G)$, as wanted in (iii). □

Lemma 2.14 *For $\alpha \in \{n, n-1\}$, let $r_\alpha \in Z_\alpha(-q)$.*

(i) $|G|_{r_\alpha} = |PSU_n(q)|_{r_\alpha}$.

(ii) *If $\gamma \in cs(G) - \{1\}$ such that $|\gamma|_{r_\alpha} < |G|_{r_\alpha}$, then there exists a divisor m of α such that $\gamma = \frac{|GU_n(q)|}{\beta(q+1)^{n-\alpha}|GL^{\epsilon}_{\alpha/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$ and either $\beta = 1$ or $\alpha = n$ and $\beta \mid \gcd(q+1, m)$. Also, if either $\alpha = n$ or $\alpha = n-1$ and $q+1 \mid n$, then $m \neq 1$.*

Proof. Corollary 2.10, Lemmas 2.11 and 2.13, and a trivial verification guarantee that r_α and $\frac{|GU_n(q)|}{(q+1)^{n-\alpha}(q^\alpha-(-1)^\alpha)}$ satisfy the assumptions of Lemma 2.12(ii) and so complete the proof of (i). Now (ii) follows from (i), Corollary 2.10 and Lemma 2.11. □

Lemma 2.15 [6, Lemma 2.12] *Let H be a finite group with $Z(H) = \{1\}$ and $r \in \pi(H)$ such that $|H|_r = \mathrm{Max}\{|\gamma|_r : \gamma \in cs(H)\}$. Let x be a non-trivial r-element of H, let $B = \{\gamma \in cs(H) - \{1\} : |\gamma|_r < |H|_r\}$ and let ξ be maximal in $cs(H)$ by divisibility. Assume $|\xi|_r = 1$ and for every $\beta \in B - \{\xi\}$, either there exists $t \in \pi(H) - \{r\}$ such that $|\xi|_t \neq 1$ and one of the following holds:*

(a) *$|\beta|_t = 1$, $|H|_t = \mathrm{Max}\{|\gamma|_t : \gamma \in cs(H)\}$ and there is not any $\delta \in B - \{\beta\}$ with $|\delta|_t < |H|_t$ and $\beta \mid \delta$;*

(b) $|\beta|_t = \mathrm{Min}\{|\gamma|_t : \gamma \in B\} \neq |H|_t, 1$,

or $B' = \{\gamma \in B : \beta \mid \gamma\}$ *contains exactly two elements and for every* $\gamma \in B'$, *we have* $|\beta|_r = |\gamma|_r$ *and either* $|\gamma| = |\beta|$ *or* $|\gamma|_{r'}/|\beta|_{r'}$ *is not a prime power. Then*

(i) $|cl_H(x)| = \xi$. *Moreover,* $C_H(x) = O_r(C_H(x)) \times O_{r'}(C_H(x))$, $O_{r'}(C_H(x))$ *is abelian and* $C_H(x)$ *is nilpotent.*

(ii) *For every* r'*-element* $w \in C_H(x)$, $C_H(x) \leq C_H(w)$.

Lemma 2.16 *For* $\alpha \in \{n, n-1\}$, *let* $r_\alpha \in Z_\alpha(-q)$. *Then*

(i) *for every* r_{n-1}*-element* $x_{n-1} \in G - \{1\}$, $|cl_G(x_{n-1})| = \frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})}$. *Moreover,* $C_G(x_{n-1}) = O_{r_{n-1}}(C_G(x_{n-1})) \times O_{r'_{n-1}}(C_G(x_{n-1}))$, $O_{r'_{n-1}}(C_G(x_{n-1}))$ *is abelian and* $C_G(x_{n-1})$ *is nilpotent.*

(ii) *If* n *is prime or* $(n, q) = (4, 2)$, *then for every* r_n*-element* $x_n \in G - \{1\}$, $|cl_G(x_n)| = \frac{|GU_n(q)|}{(q^n-(-1)^n)}$. *Moreover,* $C_G(x_n) = O_{r_n}(C_G(x_n)) \times O_{r'_n}(C_G(x_n))$, $O_{r'_n}(C_G(x_n))$ *is abelian and* $C_G(x_n)$ *is nilpotent.*

(iii) *For every* r'_{n-1}*-element* $w_{n-1} \in C_G(x_{n-1})$, $C_G(x_{n-1}) \leq C_G(w_{n-1})$.

(iv) *If* n *is prime or* $(n, q) = (4, 2)$, *then for every* r'_n*-element* $w_n \in C_G(x_n)$, $C_G(x_n) \leq C_G(w_n)$.

Proof. Fix $T_\alpha = \{\beta \in cs(G) - \{1\} : |\beta|_{r_\alpha} < |G|_{r_\alpha}\}$. Lemmas 2.13 and 2.14(ii), and a trivial verification lead us to see that r_α, $\frac{|GU_n(q)|}{(q+1)^{n-\alpha}(q^\alpha-(-1)^\alpha)}$ and T_α satisfy the assumptions of Lemma 2.15 and so complete the proof. $\square$

Lemma 2.17 *Let* $u \in \pi(PSU_n(q)) - \{p\}$.

(i) *If* $\{q, u\} \neq \{2, 3\}$, *then* $|PSU_n(q)|_u < q^{3n/2}$. *Also, if*

$$(q, u) \notin \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\},$$

then $|PSU_n(q)|_u < \frac{1}{2}q^{n-1}q^{(n-1)/4}$.

(ii) *If* $\{q,u\}\neq\{2,3\}$*, then for* $(q,u)\neq(7,2),(8,3)$, $|PSU_4(q)|_u<q^{3.5}$ *and,* $|PSU_4(7)|_2<7^{3.57}$, $|PSU_4(8)|_3<8^{3.7}$, $|PSU_4(2)|_3<2^{6.5}$ *and* $|PSU_4(3)|_2<3^{4.5}$*. If* $\{q,u\}\neq\{2,3\}$*, then* $|PSU_5(q)|_u<q^{5.5}$ *and,* $|PSU_5(2)|_3<2^8$ *and* $|PSU_5(3)|_2<3^7$*. If* $\{q,u\}\neq\{2,3\}$*, then* $|PSU_6(q)|_u<q^7$ *and,* $|PSU_6(2)|_3<2^{10}$ *and* $|PSU_6(3)|_2<3^9$*. Moreover,* $|PSU_n(3)|_2<3^{1.9n-2.4}$ *and* $|PSU_n(2)|_3<2^{2.4n-0.8}$.

(iii) *If* $n\geq 3$*, then for every* $x\in PSU_n(q)-\{1\}$*, either* $|cl_{PSU_n(q)}(x)|>|PSU_n(q)|_u$ *or* $\{q,u\}=\{2,3\}$*. Also, if* $n\geq 6$ *and* $(q,u)\notin\{(2,3),(3,2),(7,2),(8,3),(4,5)\}$*, then for every* $x\in PSU_n(q)-\{1\}$*, either* $|cl_{PSU_n(q)}(x)|_{p'}>|PSU_n(q)|_u$ *or* $q\notin\{2,3,4,7,8\}$, $q+1\nmid n$ *and* $|cl_{PSU_n(q)}(x)|=\frac{|SU_n(q)|}{|GU_{n-1}(q)|}$.

Proof. Considering the order of $PSU_n(q)$ completes the proof of (i) and (ii). Since $C_{PSU_n(q)}(x)<PSU_n(q)$, we deduce that there exists a maximal subgroup M of $PSU_n(q)$ containing $C_{PSU_n(q)}(x)$. Considering the orders of maximal subgroups of $PSU_n(q)$, mentioned in [12, Tables 3.5.A-F] and the structural properties of members of these tables [12, Chap. 4] completes the proof of (iii). □

Remark 2.18 *Let* $r_n\in Z_n(-q)$*. If* n *is an odd prime or* $(n,q)=(4,2)$*, then* $\gcd(\frac{q^n-(-1)^n}{(q+1)\gcd(n,q+1)},q+1)=1$ *and hence* Lemma 2.14(ii) *shows that*

$$\{\beta\in cs(G)-\{1\}:|\beta|_{r_n}<|PSU_n(q)|_{r_n}\}=\Big\{\frac{|GU_n(q)|}{(q^n-(-1)^n)}\Big\}. \qquad (2)$$

If there exists $t\in\pi(G)=\pi(PSU_n(q))$ *such that* $|G|_t>|PSU_n(q)|_t$*, then* Lemma 2.14(i) *shows that* $t\notin Z_n(-q)\cup Z_{n-1}(-q)$ *and* Lemma 2.13(ii) *forces* $G-\{1\}$ *to contain a* t*-element* x *such that* $|cl_G(x)|$ *divides* $\frac{|GU_n(q)|}{q^n-(-1)^n}$ *and hence,* $r_n\nmid|cl_G(x)|$*. Thus* (2) *shows that* $|cl_G(x)|=\frac{|GU_n(q)|}{(q^n-(-1)^n)}$*. Therefore,* $C_G(x)$ *contains a non-trivial* r_n*-element* w*, which by* (2), $|cl_G(x)|=|cl_G(w)|$*. So* Lemma 2.16(iv) *guarantees that* $Z(T)\leq C_G(w)=C_G(x)$*, for some* $T\in \mathrm{Syl}_t(G)$*. Thus again* Lemma 2.16(iv) *shows that if* $y\in Z(T)$*, then* $C_G(w)\leq C_G(y)$ *and hence* $|cl_G(y)|$ *divides* $|cl_G(w)|$*. Therefore,* $r_n\nmid|cl_G(y)|$*. Thus* (2) *shows that* $|cl_G(y)|=\frac{|GU_n(q)|}{(q^n-(-1)^n)}$*. But* $y\in Z(T)$*, so* $t\nmid|cl_G(y)|$ *and hence,* $t\in\pi(\frac{(q^n-(-1)^n)}{(q+1)\gcd(n,q+1)})$*, by* Lemma 2.13(iii)*. On the other hand,* n *is prime or* $(n,q)=(4,2)$ *and hence* $\pi(\frac{(q^n-(-1)^n)}{(q+1)\gcd(n,q+1)})=Z_n(-q)$*, which is a contradiction because* $t\notin Z_n(-q)$*. Thus* $|G|=|PSU_n(q)|$.

Also, let $r_{n-1}\in Z_{n-1}(-q)$*. If* $n-1$ *is prime and* $q+1\mid n$*, then* Lemma 2.14(ii) *shows* *that*

$$\{\beta \in cs(G)-\{1\} : |\beta|_{r_{n-1}} < |PSU_n(q)|_{r_{n-1}}\} = \Big\{\frac{|GU_n(q)|}{(q+1)(q^{n-1}-(-1)^{n-1})}\Big\}.$$

Thus the same reasoning as above shows that $|G| = |PSU_n(q)|$.

Lemma 2.19 [6, Lemma 2.15] *Let* H *be a finite group with* $Z(H)=1$ *and* $r,t \in \pi(H)$.

(i) *If for every* $\beta \in cs(H)-\{1\}$ *with* $|\beta|_r < |H|_r$, $t \mid \beta$, *then for every non-trivial* r*-element* $x_r \in H$ *and* $T \in \mathrm{Syl}_t(H)$, $C_H(x_r)\cap Z(T) = 1$.

(ii) *If for every* $\beta \in cs(H)-\{1\}$, *either* $|\beta|_r = |H|_r$ *or* $|\beta|_t = |H|_t$, *then*

(a) $rt \notin \omega(H)$;

(b) *for every* r*-element* $x_r \in H-\{1\}$ *and* t*-element* $x_t \in H-\{1\}$, $C_H(x_r)\cap C_H(x_t) = 1$. *In particular, for every* $u \in \pi(H)$, $|C_H(x_r)|_u \le |cl_H(x_t)|_u$ *and* $|H|_u \le |cl_H(x_r)|_u |cl_H(x_t)|_u$.

Lemma 2.20 *For some* $\pi \subseteq \pi(G)$, *let* K *be a normal* π*-subgroup of* G *and* $\bar{G} = \frac{G}{K}$. *For* $\alpha \in \{n, n-1\}$, *let* $(n,q) = (4,2)$ *and* $r_3 = r_4 = 5$ *or* $(n,q) = (3,3)$ *and* $r_2 = r_3 = 7$ *or* $(n,q) \neq (4,2),(3,3)$ *and* $r_\alpha \in Z_\alpha(-q)$. *Let* x_α *be an* r_α*-element of* $G-\{1\}$. *Then:*

(i) *for every* $P \in \mathrm{Syl}_p(G)$, $C_G(x_\alpha)\cap Z(P) = 1$. *Also, if* $\{q,t\} = \{2,3\}$ *and* $T \in \mathrm{Syl}_t(G)$, *then* $C_G(x_n)\cap Z(T) = \{1\}$;

(ii) *if* $(n,q) \neq (3,3),(4,2)$, *then for every* $\gamma \in cs(G)-\{1\}$, *either* $|\gamma|_{r_n} = |G|_{r_n}$ *or* $|\gamma|_{r_{n-1}} = |G|_{r_{n-1}}$;

(iii) *if* $(n,q) \neq (3,3),(4,2)$, *then* $r_n r_{n-1} \notin \omega(G)$;

(iv) *if* $(n,q) \neq (3,3),(4,2)$, *then* $C_G(x_n)\cap C_G(x_{n-1}) = \{1\}$;

(v) *for every* $t \in \pi(G)$, *either* $(n,q) \in \{(3,3),(4,2)\}$ *and* $|G|_t = |PSU_n(q)|_t$ *or*

$$|G|_t \le \frac{(|GU_n(q)|_t)^2}{|q+1|_t |q^n-(-1)^n|_t |q^{n-1}-(-1)^{n-1}|_t}.$$

In particular, $|G|_t \le (|PSU_n(q)|_t)^2$ *and* $|C_G(x_\alpha)|_t \le |PSU_n(q)|_t$;

(vi) *if* $r_n, r_{n-1} \notin \pi$, *then* $|\frac{(q+1)^{n-\alpha}(q^\alpha-(-1)^\alpha)|}{(q+1)\gcd(n,q+1)}|_{\pi'} \mid |C_{\bar{G}}(\bar{x}_\alpha)|$;

(vii) *if* $r_n, r_{n-1} \notin \pi$, *then* $C_{\bar{G}}(\bar{x}_{n-1})$ *is nilpotent and* $O_{r'_{n-1}}(C_{\bar{G}}(\bar{x}_{n-1}))$ *is abelian. Also, if* n *is prime or* $(n,q) = (4,2)$, *then* $C_{\bar{G}}(\bar{x}_n)$ *is nilpotent and* $O_{r'_n}(C_{\bar{G}}(\bar{x}_n))$ *is abelian.*

Proof. (i) follows immediately from Lemmas 2.14(ii) and 2.19(i). For the proof of (ii), we assume that such $\gamma \in cs(G)$ exists. We derive a contradiction to this assumption. Since $|\gamma|_{r_n} \neq |G|_{r_n}$, we deduce from Lemma 2.14(ii) that $\gamma = \frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|}$, where $m \neq 1$ is a divisor of n, $\epsilon = \mathrm{sgn}((-1)^m)$ and $\beta \mid \gcd(q+1, m)$. Thus considering Lemma 2.14(i) gives that $|\gamma|_{r_{n-1}} = |PSU_n(q)|_{r_{n-1}} = |G|_{r_{n-1}}$, which is a contradiction. From (ii) and Lemma 2.19(ii)(a,b), we obtain (iii) and (iv). Also, if $(n, q) = (3, 3), (4, 2)$, then Remark 2.18 shows that $|G| = |PSU_n(q)|$ and otherwise, by Lemma 2.19(ii)(b), for every $t \in \pi(G)$, $|G|_t \leq |cl_G(x_n)|_t |cl_G(x_{n-1})|_t$. Thus (v) follows from Lemma 2.14(ii). Now we prove (vi). From Lemmas 2.14(ii) and 2.16(i), $|PSU_n(q)|$ divides $|G|$ and $|cl_G(x_\alpha)| \mid \frac{|PSU_n(q)|(q+1)\gcd(n,q+1)}{(q+1)^{n-\alpha}(q^\alpha-(-1)^\alpha)}$. Thus $\frac{|G|(q+1)^{n-\alpha}(q^\alpha-(-1)^\alpha)}{|PSU_n(q)|(q+1)\gcd(n,q+1)} \mid |C_G(x_\alpha)|$. Also Lemma 2.4(iv) shows that $C_{\bar{G}}(\bar{x}_\alpha) = \frac{C_G(x_\alpha)K}{K} \cong \frac{C_G(x_\alpha)}{C_K(x_\alpha)}$, so (vi) follows and Lemma 2.16(i,ii) completes the proof of (vii). □

Lemma 2.21 *Let $r_n \in Z_n(-q)$ and x_n be an r_n-element of $G - \{1\}$. Also let $K \trianglelefteq G$ be a s-group for some $s \in \pi(G)$.*

(i) *If $S \in \mathrm{Syl}_s(G)$ such that $K \cap C_S(x_n) \neq \{1\}$, then there exists $1 \neq y_n \in K \cap C_S(x_n)$ such that $Z(K)C_S(x_n) \leq C_G(y_n)$.*

(ii) *If $S \in \mathrm{Syl}_s(G)$ such that $Z(K) \cap C_S(x_n) \neq \{1\}$, then there exists $1 \neq y_n \in Z(K) \cap C_S(x_n)$ such that $KC_S(x_n) \leq C_G(y_n)$.*

Proof. Since $K \trianglelefteq G$, $\{1\} \neq K \cap C_S(x_n) \trianglelefteq C_S(x_n)$ and hence, $Z(C_S(x_n)) \cap (K \cap C_S(x_n)) \neq \{1\}$. Thus there exists $1 \neq y_n \in Z(C_S(x_n)) \cap K$, so $C_S(x_n) \leq C_G(y_n)$. Also, $y_n \in K$ and hence, $Z(K) \leq C_G(y_n)$. Therefore, $Z(K)C_S(x_n) \leq C_G(y_n)$, as desired in (i). The same argument completes the proof of (ii). □

Lemma 2.22 *Let $(n, q) \neq (3, 3), (4, 2)$, $\alpha \in \{n, n-1\}$, $r_\alpha \in Z_\alpha(-q)$ and let x_α be an r_α-element of $G - \{1\}$. Also let $K \trianglelefteq G$ be an abelian s-group for some $s \in \pi(G)$. If $C_K(x_n), C_K(x_{n-1}) \neq \{1\}$, then there exist a divisor m_1 of n and a divisor m_2 of $n-1$ such that $m_1 \neq 1$ and $|K| \leq \frac{|\beta|_s |GL^{\varepsilon_1}_{n/m_1}(q^{m_1})|_s |GL^{\varepsilon_2}_{(n-1)/m_2}(q^{m_2})|_s}{|q^n-(-1)^n|_s |q^{n-1}-(-1)^{n-1}|_s}$, where β divides $\gcd(m_1, q+1)$, $\epsilon_1 = \mathrm{sgn}((-1)^{m_1})$ and $\epsilon_2 = \mathrm{sgn}((-1)^{m_2})$.*

Proof. Since $C_K(x_n) \neq \{1\}$, there exists $S \in \mathrm{Syl}_s(G)$ such that $1 \neq C_S(x_n) \in \mathrm{Syl}_s(C_G(x_n))$, so Lemma 2.21 shows that there exists $1 \neq y_n \in C_K(x_n)$ such that $Z(K)C_S(x_n) = KC_S(x_n) \leq$

$C_G(y_n)$. Also, if $1 \neq y_{n-1} \in C_K(x_{n-1})$, then Lemma 2.16(iii) shows that $KC_G(x_{n-1}) \leq C_G(y_{n-1})$. Therefore, $|cl_K(x_n)| = \frac{|K|}{|C_K(x_n)|}$ divides $\frac{|C_G(y_n)|_s}{|C_S(x_n)|} = \frac{|cl_G(x_n)|_s}{|cl_G(y_n)|_s}$ and $|cl_K(x_{n-1})| = \frac{|K|}{|C_K(x_{n-1})|}$ divides $\frac{|C_G(y_{n-1})|_s}{|C_G(x_{n-1})|_s} = \frac{|cl_G(x_{n-1})|_s}{|cl_G(y_{n-1})|_s}$. On the other hand, Lemma 2.14(ii) implies that there exist a divisor m_1 of n and a divisor m_2 of $n-1$ such that $m_1 \neq 1$, $\frac{|cl_G(x_n)|_s}{|cl_G(y_n)|_s}$ divides $\frac{|\beta|_s |GL^{\varepsilon_1}_{n/m_1}(q^{m_1})|_s}{|q^n-(-1)^n|_s}$ and $\frac{|cl_G(x_{n-1})|_s}{|cl_G(y_{n-1})|_s}$ divides $\frac{|GL^{\varepsilon_2}_{(n-1)/m_2}(q^{m_2})|_s}{|q^{n-1}-(-1)^{n-1}|_s}$, where $\beta \mid \gcd(m_1, q+1)$, $\epsilon_1 = \mathrm{sgn}((-1)^{m_1})$ and $\epsilon_2 = \mathrm{sgn}((-1)^{m_2})$. Since $C_K(x_n)C_K(x_{n-1}) \leq K$ and $C_K(x_n) \cap C_K(x_{n-1}) = \{1\}$, by Lemma 2.20(iv), $|C_K(x_n)|$ divides $\frac{|K|}{|C_K(x_{n-1})|}$. Therefore, $|K| = |C_K(x_n)||cl_K(x_n)|$ divides $|cl_K(x_{n-1})||cl_K(x_n)|$, hence the above statements complete the proof. □

Lemma 2.23 *Let H be a finite simple group of Lie type over a field with q elements such that $|H|_p = q^u$. If $r \in \pi(H) - \{p\}$, then there exists $1 \leq i \leq u$ such that $r \in Z_i(q)$ unless*

(i) $H = PSL_2(q)$ *and* $r \in Z_2(q)$;

(ii) $H = PSU_3(q)$ *and* $r \in Z_6(q)$;

(iii) $H = {}^2B_2(q)$ *and* $r \in Z_4(q)$;

(iv) $H = {}^2G_2(q)$ *and* $r \in Z_6(q)$.

Proof. The proof follows immediately by considering the orders of finite simple groups of Lie type. □

Lemma 2.24 [2, Proof of Theorem 3.3, Case 2] *Let $r \in Z_n(q)$. If w is a non-trivial r-element of $PSL_n(q)$ and ψ is a non-trivial field automorphism of $PSL_n(q)$, then $PSL_n(q)$ does not contain any element g such that $(\psi i_g)^{-1} i_w (\psi i_g) \in \{i_w, i_{(w^t)^{-1}}\}$, where for every $x, y \in PSL_n(q)$, $i_y(x) = y^{-1}xy$.*

Theorem 2.25 *If $N = PSU_n(q) \trianglelefteq H \leq \mathrm{Aut}(PSU_n(q))$ and $cs(H) = cs(PSU_n(q))$, then $H \cong PSU_n(q)$.*

Proof. Let $\mathbf{0}$ be a column vector with entries 0 and $\mathbf{1}$ be a column vector with entries 1. Let $J_1 = A_1 = (1)$, $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$, for some $a_2 \in GF(q^2) - \{0\}$ such that $a_2 + a_2^q = 0$, where $GF(q^2)$ denotes a field with q^2 elements. For $n \geq 3$, fix $J_n = \begin{pmatrix} 0 & \mathbf{0}^t & 1 \\ \mathbf{0} & J_{n-2} & \mathbf{0} \\ 1 & \mathbf{0}^t & 0 \end{pmatrix}$ and $A_n = \begin{pmatrix} 1 & -\mathbf{1}^t A_{n-2} & a_n \\ \mathbf{0} & A_{n-2} & \mathbf{1} \\ 0 & \mathbf{0}^t & 1 \end{pmatrix}$, for some $a_n \in GF(q^2)$ with $a_n + a_n^q = -\mathbf{1}^t J_{n-2} \mathbf{1}$.

Since $SU_n(q) = \{A \in SL_n(q^2) : A^t J_n A^\tau = J_n\}$, we get that $A_n \in SU_n(q)$. Note that for a diagonal automorphism δ of $PSU_n(q)$ of order $\gcd(n, q+1)$, $PSU_n(q).\langle\delta\rangle \cong PGU_n(q)$ and an easy calculation shows that $|C_{PGU_n(q)}(A_n Z(GU_n(q)))|$ is a p-number. Thus if H contains a non-trivial diagonal automorphism, then $|C_{H\cap(PSU_n(q).\langle\delta\rangle)}(\bar{A}_n)|$ is a p-number and hence, for some $s \in \pi(H \cap (PSU_n(q).\langle\delta\rangle)/PSU_n(q))$, $|cl_H(\bar{A}_n)|_s > |PSU_n(q)|_s$, where $\bar{A}_n$ is the image of A_n in H. Therefore, $|cl_H(\bar{A}_n)| \in cs(H) - cs(PSU_n(q))$. So $cs(H) \neq cs(PSU_n(q))$, which is a contradiction. This shows that H does not contain any diagonal automorphism of $PSU_n(q)$.

Now let H contain a field automorphism ψ. If n is odd, then let $r \in Z_n(-q)$ and let A be a non-trivial r-element of $PSU_n(q)$. An easy verification shows that $Z_n(-q) \subseteq Z_n(q^2)$, so $r \in Z_n(q^2)$. Since $PSU_n(q) \lesssim PSL_n(q^2)$, Lemma 2.24 shows that $C_{PSU_n(q).\langle\psi\rangle}(i_A) = C_{PSU_n(q)}(i_A)$, where for every $x \in PSU_n(q)$, $i_A(x) = A^{-1}xA$. Also, it is known that $|C_{PSU_n(q)}(i_A)| = \frac{(q^n+1)}{(q+1)\gcd(n,q+1)}$. Therefore, for some divisors $k' \neq 1$ of k and k'' of $\gcd(n, q+1)$, $|cl_H(i_A)| = \frac{k'k''|GU_n(q)|}{(q^n+1)}$, which is a contradiction because by Lemma 2.13(i), $\frac{|GU_n(q)|}{(q^n+1)}$ is maximal in $cs(PSU_n(q))$ by divisibility. Now let n be even and $r \in Z_n(-q)$. Again an easy verification shows that $Z_n(-q) \subseteq Z_{n/2}(q^2)$ and hence, $r \in Z_{n/2}(q^2)$. Let A be a non-trivial r-element of $SL_{n/2}(q^2)$. Then since $SU_n(q) = \{C \in SL_n(q^2) : C^t K_n C^\tau = K_n\}$, where $I_n = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n-\text{times}})$ and $K_n = \begin{pmatrix} \mathbf{0} & I_{n/2} \\ I_{n/2} & \mathbf{0} \end{pmatrix}$, we have $E = \begin{pmatrix} (A^t)^{-1} & \mathbf{0} \\ \mathbf{0} & A^\tau \end{pmatrix} Z(SU_n(q))$ is an r-element of $PSU_n(q)$ and hence, by considering Lemma 2.24, we see that $C_{PSU_n(q).\langle\psi\rangle}(i_E) = C_{PSU_n(q)}(i_E)$. Thus applying the above argument leads us to a contradiction.

This shows that H does not contain any field automorphism of $PSU_n(q)$. The same reasoning shows that H does not contain any diagonal-field automorphism. Thus $H \cong PSU_n(q)$, as claimed. $\square$

3 The proof of the main theorem

By assumption, $n \geq 3$ and since $PSU_n(q)$ is considered as a simple group, $(n, q) \neq (3, 2)$. Define the natural function τ as follows:

$$\tau(m) = \begin{cases} m, & \text{if } m \text{ and } m/2 \text{ are even} \\ m/2, & \text{if } m \text{ is even and } m/2 \text{ is odd} \\ 2m, & \text{if } m \text{ is odd} \end{cases}.$$

Since $q = p^k$, for every natural number m, $Z_{\tau(m)k}(p) \subseteq Z_m(-q)$ and by Lemma 2.2, $Z_{\tau(m)k}(p) = \emptyset$ if and only if $(m, q) \in \{(3, 2), (2, 3), (2, 2)\}$. Thus $Z_{\tau(n)k}(p) \neq \emptyset$ and also, $Z_{\tau(n-1)k}(p) = \emptyset$ if and only if $(n, q) \in \{(4, 2), (3, 3)\}$. So hereafter, we may assume $r_n \in Z_{\tau(n)k}(p) \subseteq Z_n(-q)$. Also, if $(n, q) \neq (4, 2), (3, 3)$, let $r_{n-1} \in Z_{\tau(n-1)k}(p) \subseteq Z_{n-1}(-q)$ and otherwise, let $r_{n-1} = r_n$. For $\alpha \in \{n, n-1\}$, suppose that x_α is an r_α-element of $G - \{1\}$ and let N be a normal subgroup of G such that for some $s \in \pi(G)$, N is s-elementary abelian and $|N| = s^e$. We prove that $N = 1$. Suppose by contradiction that $N \neq 1$ and hence, $O_s(G) \neq 1$. Since N is a normal and abelian subgroup of G, we deduce that for every $y \in N - \{1\}$,

$$cl_G(y) \subset N \leq C_G(y). \tag{3}$$

Therefore,

$$|cl_G(y)| < |N| \leq |C_G(y)|_s \leq |G|_s. \tag{4}$$

Let $N = \Omega_1(O_s(G))$, then

$$|cl_G(y)| < |O_s(G)| \leq |G|_s. \tag{5}$$

We prove the main theorem in a sequence of steps.

Step 1. If n is prime or $(n, q) = (4, 2)$, then $O_s(G) \cap C_G(x_n) = \{1\}$. Moreover, if $n - 1$ is prime and $q + 1 \mid n$, then $O_s(G) \cap C_G(x_{n-1}) = \{1\}$.

Proof. Let n be a prime or $(n, q) = (4, 2)$ and let $1 \neq y_n \in O_s(G) \cap C_G(x_n)$. By Remark 2.18, $|G| = |PSU_n(q)|$ and

$$|cl_G(y_n)| \in \{\gamma \in cs(G) - \{1\} : |\gamma|_{r_n} < |PSU_n(q)|_{r_n}\} = \Big\{\frac{|GU_n(q)|}{(q^n - (-1)^n)}\Big\}.$$

Also, by (5), $q^{n(n-1)/2+(n-1)} < \frac{|GU_n(q)|}{(q^n-(-1)^n)} = |cl_G(y_n)| < |O_s(G)| \leq |G|_s$ and either $n \neq 3$ and $|G|_s = |PSU_n(q)|_s < q^{\max\{n(n-1)/2, 2.4n-0.8\}}$ or $|G|_s = |PSU_n(q)|_s < q^5$, which is impossible. So $O_s(G) \cap C_G(x_n) = \{1\}$.

Let $n - 1$ be prime and $q + 1 \mid n$. If $1 \neq y_{n-1} \in O_s(G) \cap C_G(x_{n-1})$, then replacing r_n and $\frac{|GU_n(q)|}{(q^n-(-1)^n)}$ with r_{n-1} and $\frac{|GU_n(q)|}{(q^{n-1}-(-1)^{n-1})(q+1)}$ in the above statement completes the proof. □

Step 2. If $s \neq p$, then $O_s(G) \cap C_G(x_n) = \{1\}$ and if $s = p$, then $O_s(G) \cap C_G(x_{n-1}) \neq \{1\}$. In particular, $Z(O_p(G)) \cap C_G(x_{n-1}) \neq \{1\}$ and if $1 \neq y_{n-1} \in Z(O_p(G)) \cap C_G(x_{n-1})$, then n is not prime, $q + 1 \nmid n$ and $|cl_G(y_{n-1})| = \frac{|GU_n(q)|}{(q+1)|GU_{n-1}(q)|}$.

Proof. On the contrary, let $s \neq p$ and $1 \neq y_n \in O_s(G) \cap C_G(x_n)$. Thus there exists a divisor m of n such that $m \neq 1$ and $|cl_G(y_n)| = \frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$ and $\beta \mid \gcd(m, q+1)$. Also by (5), $|cl_G(y_n)| < |O_s(G)| \leq |G|_s$. Thus Lemma 2.20(v) shows that $\frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|} \leq \frac{(|GU_n(q)|_s)^2}{|q^n-(-1)^n|_s|q+1|_s|q^{n-1}-(-1)^{n-1}|_s}$, which is impossible. Therefore, $O_s(G) \cap C_G(x_n) = \{1\}$, as wanted.

Now let $s = p$. Suppose by contradiction that $O_p(G) \cap C_G(x_{n-1}) = \{1\}$. Thus $|O_p(G)| \leq |cl_G(x_{n-1})|_p \leq |PSU_n(q)|_p$, by Lemma 2.4(ii). If $1 \neq y \in O_p(G) \cap C_G(x_n)$, then by Lemma 2.14(ii) and (5), there exists a divisor m of n such that $m \neq 1$ and $|cl_G(y)| = \frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$ and $\beta \mid \gcd(m, q+1)$, and $|cl_G(y)| < |O_p(G)|$. Therefore, $\frac{|GU_n(q)|}{\beta|GL^{\epsilon}_{n/m}(q^m)|} < |PSU_n(q)|_p$, which is impossible. So $O_p(G) \cap C_G(x_n) = \{1\}$ and Lemma 2.4(v) forces $r_{n-1}, r_n \mid |O_p(G)| - 1 = p^a - 1$. Thus $\tau(n)k, \tau(n-1)k \mid a$. This shows that $n(n-1)k \mid a$, which is impossible because $p^a = |O_p(G)| \leq |PSU_n(q)|_p = p^{n(n-1)k/2}$. Therefore, $O_p(G) \cap C_G(x_{n-1}) \neq \{1\}$, as claimed. The same reasoning shows that $Z(O_p(G)) \cap C_G(x_{n-1}) \neq \{1\}$.

If $Z(O_p(G)) \cap C_G(x_{n-1}) \neq \{1\}$ and n is prime, then since Remark 2.18 shows that $|G| = |PSU_n(q)|$ and $|cl_G(x_{n-1})|_p = |PSU_n(q)|_p$, we get that $|C_G(x_{n-1})|_p = 1$, which is a contradiction. So if $O_p(G) \neq \{1\}$, then n is not prime.

Finally suppose, contrary to our claim, that $1 \neq y_{n-1} \in Z(O_p(G)) \cap C_G(x_{n-1})$ such that $|cl_G(y_{n-1})| \neq \frac{|GU_n(q)|}{(q+1)|GU_{(n-1)}(q)|}$. Lemma 2.14(ii) shows that there exists a divisor $m \neq 1$ of $n-1$ such that $|cl_G(y_{n-1})| = \frac{|GU_n(q)|}{(q+1)|GL^{\epsilon}_{(n-1)/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$. If $Z(O_p(G)) \cap C_G(x_n) = \{1\}$, then Lemma 2.4(ii) shows that $|Z(O_p(G))| < |cl_G(x_n)|_p \leq |PSU_n(q)|_p = q^{n(n-1)/2}$. If $Z(O_p(G)) \cap C_G(x_n) \neq \{1\}$, then applying Lemma 2.22 leads us to divisor $m_1 \neq 1$ of n such that $|Z(O_p(G))| < q^{\frac{(n-1)((n-1)/m-1)}{2}} q^{\frac{n(n/m_1-1)}{2}}$. On the other hand, $y_{n-1} \in Z(O_p(G))$, so $|cl_G(y_{n-1})| < |Z(O_p(G))|$ and hence, $\frac{|GU_n(q)|}{(q+1)|GL^{\epsilon}_{(n-1)/m}(q^m)|} < q^{n(n-1)/2}$ or $\frac{|GU_n(q)|}{(q+1)|GL^{\epsilon}_{(n-1)/m}(q^m)|} < q^{\frac{(n-1)((n-1)/m-1)}{2}} q^{\frac{n(n/m_1-1)}{2}}$, which it is impossible. □

Step 3. Let $N \neq \{1\}$. Then $n \geq 9$ and $\{q, s\} = \{2, 3\}$ or $n \geq 6$, $s = p$, n is not prime and $q + 1 \nmid n$. If $s = p$ and $n = 6$, then $O_p(G) \cap C_G(x_n) = \{1\}$.

Proof. Let $s \neq p$. By Step 2, $N \cap C_G(x_n) = \{1\}$. Lemma 2.4(ii) and (4) show that for every $y \in G$, $|cl_G(y)| < |N| \leq |cl_G(x_n)|_s \leq |PSU_n(q)|_s$. Lemma 2.17(iii) gives that $\{q, s\} = \{2, 3\}$.

Now let $n = 8$. If $q = 3$ and $s = 2$, then $|N| \leq |cl_G(x_n)|_2 \leq 2^{18}$. Since $q + 1 \mid n$ and $n - 1$ is prime, Step 1 shows that $N \cap C_G(x_{n-1}) = \{1\}$ and hence, $\langle x_{n-1} \rangle$ acts fixed-point-freely on $N - \{1\}$. Thus $r_{n-1} = O(x_{n-1})$ divides $|N| - 1$. But $r_{n-1} = 547$ and $\exp_{547}(2) > 19$, which is a contradiction. Now let $q = 2$ and $s = 3$. Then $|N| \leq |cl_G(x_n)|_3 \leq 3^9$. Since $N \cap C_G(x_n) = \{1\}$, $\langle x_n \rangle$ acts fixed-point-freely on $N - \{1\}$. Thus $17 = r_n = O(x_n)$ divides $|N| - 1$. But $\exp_{17}(3) > 9$, which is a contradiction. Thus if $n = 8$, then $\{q, s\} \neq \{2, 3\}$. The same reasoning shows that if $n \in \{6, 7\}$, then $\{q, s\} \neq \{2, 3\}$; if $n = 5$, $(q, s) \neq (3, 2)$; and if $n = 4$, $(q, s) \neq (2, 3)$. If $n = 5$ and $(q, s) = (2, 3)$, then since $|N| \leq |cl_G(x_n)|_3 \leq 3^5$, for ever $y \in N$, $|cl_G(y)| < |N| \leq 243$, by (4). Therefore, considering the elements of $cs(G)$ shows that for every $y \in N - \{1\}$, $|cl_G(y)| \in \{165, 176\}$, so for some $l, h \in \mathbb{N} \cup \{0\}$ and $a \leq 5$, $165l + 176h = |N| - 1 = 3^a - 1$, which is impossible. Thus if $n = 5$, then $(q, s) \neq (2, 3)$. The same reasoning shows that if $n \in \{3, 4\}$, then $(q, s) \neq (3, 2)$, as desired.

If $s = p$, then Step 2 shows that n is not prime and $q + 1 \nmid n$. So $n \neq 3, 5$.

Now let $n = 4$, $s = p$ and $O_p(G) \neq \{1\}$. Step 2 shows that there exists $1 \neq y_{n-1} \in C_G(x_{n-1}) \cap O_p(G)$. Thus since $Z(O_p(G))C_G(x_{n-1}) \leq C_G(y_{n-1})$, $r_{n-1} \mid |Z(O_p(G))|/|C_{Z(O_p(G))}(x_{n-1})| = p^e$ and $|Z(O_p(G))|/|C_{Z(O_p(G))}(x_{n-1})|$ divides $\frac{|C_G(y_{n-1})|_p}{|C_G(x_{n-1})|_p} = \frac{|cl_G(x_{n-1})|_p}{|cl_G(y_{n-1})|_p}$. Also, $|cl_G(y_{n-1})| \in \left\{\frac{|GU_4(q)|}{(q+1)(q^3+1)}, \frac{|GU_4(q)|}{(q+1)|GU_3(q)|}\right\}$, so $6k \mid e$ and $p^e \leq \frac{|C_G(y_{n-1})|_p}{|C_G(x_{n-1})|_p} = \frac{|cl_G(x_{n-1})|_p}{|cl_G(y_{n-1})|_p} \leq q^3$. This shows that $e = 0$ and hence, $Z(O_p(G)) \leq C_G(x_{n-1})$. Thus for $Q \in \mathrm{Syl}_p(G)$, $\{1\} \neq Z(O_p(G)) \cap Z(Q) \leq C_G(x_{n-1}) \cap Z(Q)$, so $Z(Q) \cap C_G(x_{n-1}) \neq \{1\}$, which is a contradiction to Lemma 2.20(i). This forces $O_p(G) = \{1\}$, as wanted.

Our next concern is the case $n = 6$ and $s = p$. If $Z(O_p(G)) \cap C_G(x_n) \neq \{1\}$, then there exists $1 \neq y_n \in Z(O_p(G)) \cap C_G(x_n)$ such that for some $P \in \mathrm{Syl}_p(C_G(y_n))$, $Z(O_p(G))C_P(x_n) \leq C_G(y_n)$ and $C_P(x_n) \in \mathrm{Syl}_p(C_G(x_n))$, by Lemma 2.21(ii). So there exist $m \in \{2, 3, 6\}$ and a divisor β of $\gcd(m, q + 1)$ such that $|cl_G(y_n)| = \frac{|GU_6(q)|}{\beta |GL^{\epsilon}_{6/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$. Thus $|Z(O_p(G))/C_{Z(O_p(G))}(x_n)| = p^b$ divides $|C_G(y_n)|_p/|C_G(x_n)|_p = |cl_G(x_n)|_p/|cl_G(y_n)|_p \leq |GL^{\epsilon}_{6/m}(q^m)|_p = p^a$, so $b \leq a \leq 6k$. Also, Lemma 2.4(v) shows that $r_n \mid p^b - 1$. But $\exp_{r_n}(p) = 3k$ and hence $3k \mid b$. This forces $b \in \{0, 3k, 6k\}$. On the other hand, for $Q \in \mathrm{Syl}_p(G)$, $Z(Q) \cap Z(O_p(G)) \neq \{1\}$ and $C_G(x_n) \cap Z(Q) = \{1\}$. Hence, $Z(O_p(G)) \nleq C_G(x_n)$. This shows that $b \neq 0$, so $b \in \{3k, 6k\}$ and $m \in \{2, 3\}$. By step 2, $Z(O_p(G)) \cap C_G(x_{n-1}) \neq \{1\}$. We have $C_{Z(O_p(G))}(x_n)C_{Z(O_p(G))}(x_{n-1}) \leq Z(O_p(G))$ and hence, $|C_{Z(O_p(G))}(x_n)| = p^e$ divides $|Z(O_p(G))/C_{Z(O_p(G))}(x_{n-1})| = p^f$, $p^f \leq |cl_G(x_{n-1})|_p = q^{15}$ and $r_5 \mid p^f - 1$. Thus $e \leq f \leq 10$ and hence, $q^{17} < |cl_G(y_n)| < |Z(O_p(G))| = p^b.p^e \leq q^{16}$, which is a contradiction. This shows that $Z(O_p(G)) \cap C_G(x_n) = \{1\}$. If $O_p(G) \cap C_G(x_n) \neq \{1\}$, then Lemma 2.21(i) allows us to assume that there exist $z_n \in O_p(G) \cap C_G(x_n)$ and $P \in \mathrm{Syl}_p(G)$ such that $C_P(x_n) \in \mathrm{Syl}_p(C_G(x_n))$

and $C_P(x_n) \leq C_P(z_n)$. Hence $Z(O_p(G))C_P(x_n) \leq C_G(z_n)$. By repeating the above argument, $|Z(O_p(G))| \leq q^6$. On the other hand $r_{n-1} \mid |Z(O_p(G))/C_{Z(O_p(G))}(x_{n-1})| - 1 = p^g - 1$ and hence $10k \mid g$. Therefore, $g = 0$. So $Z(O_p(G)) \leq C_G(x_{n-1})$, which is a contradiction with Lemma 2.20(i). Thus $O_p(G) \cap C_G(x_n) = \{1\}$, as wanted. $\square$

In the following, let $K_0 = O_s(G)$, where $n \geq 9$ and $\{q, s\} = \{2, 3\}$ or $n \geq 6$, $s = p$, n is not prime and $q + 1 \nmid n$. Otherwise, $K_0 = \{1\}$. Also, suppose that $\bar{M}_0 = \frac{M_0}{K_0}$ is a minimal normal subgroup of $\bar{G} = \frac{G}{K_0}$ and for every $x \in G$, let $\bar{x}$ be the image of x in $\bar{G}$.

Step 4. If $K_0 \neq \{1\}$ and $\bar{M}_0$ is a t-elementary abelian group for some $t \in \pi(G)$, then $\bar{M}_0 \cap C_{\bar{G}}(\bar{x}_n) = \{1\}$.

Proof. Suppose that, to the contrary, there exists $1 \neq \bar{y}_n \in \bar{M}_0 \cap C_{\bar{G}}(\bar{x}_n)$. So we can assume that y_n is a t-element of $C_G(x_n)$. Therefore, Lemma 2.14(ii) shows that there exist a divisor m of n and a divisor β of $\gcd(m, q+1)$ such that $|cl_G(y_n)| = \frac{|GU_n(q)|}{\beta |GL^{\epsilon}_{n/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$. Note that $K_0 = O_s(G)$, so $s \neq t$. Since y_n is a t-element, Lemmas 2.4(iv,vi) and 2.20(v), and the same reasoning given for (3) yield that

$$\begin{aligned} |cl_G(y_n)|_{s'} &\leq |cl_{\bar{G}}(\bar{y}_n)| < |\bar{M}_0| \qquad (6) \\ &\leq |C_{\bar{G}}(\bar{y}_n)|_t = |C_G(y_n)|_t = \frac{|G|_t}{|cl_G(y_n)|_t} \leq \frac{|PSU_n(q)|_t |\beta|_t |GL^{\epsilon}_{n/m}(q^m)|_t}{|q+1|_t |\gcd(n, q+1)|_t}, \end{aligned}$$

because by Lemma 2.20(v), $|G|_t \leq (|PSU_n(q)|_t)^2$. So by considering the different values of n, m and s, and Lemma 2.17(i,ii), we see that one of the following possibilities occurs:

(I) $s = p$, $(q, t) \in \{(3, 2), (4, 5), (7, 2)\}$ and $(n, m) = (6, 2)$. If $(q, t) = (3, 2)$, then (6) shows that $|\bar{M}_0| < \frac{|PSU_6(3)|_2 |\beta|_2 |GL_3(9)|_2}{4.2} \leq 2^{17}$. Since $\langle \bar{x}_5 \rangle$ acts on $\bar{M}_0$, applying Lemma 2.4(v) shows that $61 = r_5 = O(\bar{x}_5)$ divides $\frac{|\bar{M}_0|}{|C_{\bar{M}_0}(\bar{x}_5)|} - 1 = 2^{\alpha} - 1$, where $2^{\alpha} \leq |\bar{M}_0|_2 < 2^{17}$. But $\exp_{61}(2) > 17$ and hence, $\alpha = 0$. Therefore, $C_{\bar{M}_0}(\bar{x}_5) = \bar{M}_0$. So $\bar{M}_0 \leq C_{\bar{G}}(\bar{x}_5)$. This gives that $|\bar{M}_0| \leq |C_{\bar{G}}(\bar{x}_5)|_2 = |C_G(x_5)|_2 \leq |PSU_6(3)|_2$ and hence, by (6), $|cl_G(y_n)|_{p'} < |PSU_6(3)|_2$, which is impossible. The same reasoning rules out the case $(q, t) \in \{(4, 5), (7, 2)\}$.

(II) $s = p$, $(q, t) = (2, 3)$ and $(n, m) \in \{(10, 2), (8, 2)\}$. If $n = 10$ and $m = 2$, then (6) shows that $|\bar{M}_0| < |PSU_{10}(2)|_3 |GU_5(4)|_3 \leq 3^{18}$. Since $\langle \bar{x}_{10} \rangle$ acts on $\bar{M}_0$, applying Lemma 2.4(v) shows that $31 = r_{10} = O(\bar{x}_{10})$ divides $\frac{|\bar{M}_0|}{|C_{\bar{M}_0}(\bar{x}_{10})|} - 1 = 3^{\alpha} - 1$, where $3^{\alpha} \leq |\bar{M}_0|_3 < 3^{18}$. On the other hand, $\exp_{31}(3) = 30$ and hence, $\alpha = 0$. This gives $C_{\bar{M}_0}(\bar{x}_{10}) = \bar{M}_0$, so $\bar{M}_0 \leq C_{\bar{G}}(\bar{x}_{10})$. Therefore, $|\bar{M}_0| \leq |C_G(x_{10})|_3 \leq |PSU_{10}(2)|_3$ and hence, by (6), $|cl_G(y_n)|_{p'} < |PSU_{10}(2)|_3$, which is impossible. The same reasoning rules out $n = 8$ and $m = 2$.

Step 5. If $K_0 \neq \{1\}$ and $\bar{M}_0$ is a t-elementary abelian group for some $t \in \pi(G) - \{s\}$, then $n \geq 9$, $\{q, s\} = \{2, 3\}$ and $t = p$ or $n \geq 8$, $s = p$, $(q, t) \in \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\}$, n is not prime and $q + 1 \nmid n$.

Proof. Since $K_0 \neq \{1\}$, Step 3 shows that $n \geq 9$ and $\{q, s\} = \{2, 3\}$ or $n \geq 6$, $s = p$, n is not prime and $q + 1 \nmid n$. Let $\{q, s\} = \{2, 3\}$. By Steps 3 and 4, $K_0 \cap C_G(x_n) = \{1\}$ and $\bar{M}_0 \cap C_{\bar{G}}(\bar{x}_n) = \{1\}$. Thus $\langle x_n \rangle$ acts fixed-point-freely on $M_0 - \{1\}$. So M_0 is nilpotent and hence, $O_t(G) \neq 1$. Therefore, Step 3 forces $t = p$, as wanted. The same reasoning shows that if $n = 6$ and $s = p$, then $O_t(G) \neq 1$, which is impossible by considering Step 3.

Now let $s = p$. Then $t \neq p$ and by Step 4, $\bar{M}_0 \cap C_{\bar{G}}(\bar{x}_n) = \{1\}$. Thus $|\bar{M}_0| \leq |cl_{\bar{G}}(\bar{x}_n)|_t = |cl_G(x_n)|_t \leq |PSU_n(q)|_t$, by Lemma 2.4(ii,iv). So for some t-element $1 \neq y \in M_0$, Lemma 2.4(vi) yields $|cl_G(y)|_{p'} \leq |cl_{\bar{G}}(\bar{y})| < |\bar{M}_0| \leq |PSU_n(q)|_t$. Thus Lemma 2.17(iii) shows that either $q + 1 \nmid n$ and $|cl_G(y)| = \frac{|SU_n(q)|}{|GU_{n-1}(q)|}$ or $(q, t) \in \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\}$. So if $(q, t) \notin \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\}$, then $|cl_G(y)| = \frac{|SU_n(q)|}{|GU_{n-1}(q)|}$ and hence, we can assume that $y \in C_G(x_{n-1})$. On the other hand, Step 2 shows that $Z(K_0) \cap C_G(x_{n-1})$ contains a non-trivial element z such that $|cl_G(z)| = \frac{|SU_n(q)|}{|GU_{n-1}(q)|}$. Since $C_G(x_{n-1}) = O_{r_{n-1}}(C_G(x_{n-1})) \times O_{r'_{n-1}}(C_G(x_{n-1}))$, by Lemma 2.16(i), we can assume that $y, z \in O_{r'_{n-1}}(C_G(x_{n-1}))$. But $O_{r'_{n-1}}(C_G(x_{n-1}))$ is abelian, by Lemma 2.16(i), so $yz = zy$. Also $\gcd(O(y), O(z)) = \gcd(p, t) = 1$. Thus Lemma 2.4(i) shows that

$$\begin{aligned} |cl_G(yz)| &= \frac{|G|}{|C_G(y) \cap C_G(z)|} = \frac{|G||C_G(y)C_G(z)|}{|C_G(y)||C_G(z)|} \\ &\leq \frac{|G|^2}{|C_G(y)||C_G(z)|} = |cl_G(y)||cl_G(z)| = (\frac{|SU_n(q)|}{|GU_{n-1}(q)|})^2 \leq q^{4n}. \end{aligned} \quad (7)$$

On other hand, $yz \in C_G(x_{n-1})$ and hence there exists a divisor m of $n-1$ such that $|cl_G(yz)| = \frac{|SU_n(q)|}{|GL^{\varepsilon}_{(n-1)/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$. So by (7), $\frac{|SU_n(q)|}{|GL^{\varepsilon}_{(n-1)/m}(q^m)|} < q^{4n}$. This forces $m = 1$ and hence, $|cl_G(yz)| = |cl_G(y)| = |cl_G(z)|$. It follows from Lemma 2.4(i) that $C_G(y) = C_G(yz) = C_G(z)$. This shows that $K_0 \leq C_G(y)$ and hence, $1 \neq y \in C_G(K_0)$. Thus $O_t(C_G(K_0)) \neq 1$ and hence, $O_t(G) \neq 1$. So Step 3 shows that $\{q, t\} = \{2, 3\}$, which is a contradiction to our assumption. This yields that $(q, t) \in \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\}$, as wanted. □

Step 6. If $K_0 \neq \{1\}$ and there exists $t \in \pi(G)$ such that $O_t(\bar{G}) \neq \{1\}$, then $n \geq 9$, $\{q, s\} = \{2, 3\}$ and $t = p$ or $n \geq 8$, $s = p$, $(q, t) \in \{(2, 3), (3, 2), (7, 2), (8, 3), (4, 5)\}$, n is not prime and $q + 1 \nmid n$.

Proof. It follows immediately from Steps 3 and 5. □

In the following, let $n \geq 8$ and, if $q \in \{3,7\}$, fix $\pi = \{2,p\}$, if $q \in \{2,8\}$, fix $\pi = \{2,3\}$ and if $q = 4$, fix $\pi = \{2,5\}$. Otherwise, fix $\pi = \{p\}$. Let K be a maximal normal π-subgroup of G. Also, let $\bar{G} = G/K$, let $\bar{M} = M/K$ be a minimal normal subgroup of $\bar{G}$ and for every $x \in G$, let $\bar{x}$ be the image of x in $\bar{G}$.

Step 7. $\bar{M}$ is not abelian.

Proof. On the contrary suppose that $\bar{M}$ is u-elementary abelian for some $u \in \pi(G)$. So $u \notin \pi$. If $O_\pi(G) = 1$, $O_\pi(G) = O_p(G)$ or $\{q,s\} = \{2,3\}$ and $O_\pi(G) = O_s(G)$, then Steps 3 and 6 complete the proof. So let $|\pi| \geq 2$. Therefore, $n \geq 8$,

$$(q,\pi) \in \{(3,\{2,3\}),(2,\{2,3\}),(7,\{2,,7\}),(8,\{2,3\}),(4,\{2,5\})\} \tag{8}$$

and $u \notin \pi$. If $1 \neq \bar{w}_n \in \bar{M} \cap C_{\bar{G}}(\bar{x}_n)$, then we can assume that w_n is a u-element of $C_G(x_n)$. Therefore, Lemma 2.14(ii) shows that there exist a divisor $1 \neq m$ of n and a divisor β of $\gcd(m,q+1)$ such that $|cl_G(w_n)| = \frac{|GU_n(q)|}{\beta|GL^\epsilon_{n/m}(q^m)|}$, where $\epsilon = \mathrm{sgn}((-1)^m)$. Thus since w_n is a u-element, $u \notin \pi$ and $\bar{M}$ is an abelian u-group, Lemma 2.4(iv,vi) and the same reasoning given for (3) yield that

$$\begin{aligned} |cl_G(w_n)|_{\pi'} &\leq |cl_{\bar{G}}(\bar{w}_n)| < |\bar{M}| \\ &\leq |C_{\bar{G}}(\bar{w}_n)|_u = |C_G(w_n)|_u. \end{aligned} \tag{9}$$

On the other hand, Lemmas 2.14 and 2.20(v) imply that if $u \in \{r_n, r_{n-1}\}$, then $|G|_u = |PSU_n(q)|_u$ and otherwise, $|C_G(w_n)|_u = \frac{|G|_u}{|cl_G(w_n)|_u} \leq \frac{|PSU_n(q)|_u|\beta|_u|GL^\epsilon_{n/m}(q^m)|_u}{|q+1|_u|\gcd(n,q+1)|_u}$, because by Lemma 2.20(v), $|G|_u \leq (|PSU_n(q)|_u)^2$. Thus considering (9) and the different values of n, m, q and π forces $q = 2$, $\pi = \{2,3\}$, $n = 8$, $m = 2$ and $u = 5$. Applying the same argument as that used in the proof of Lemma 2.21 allows us to assume that $\bar{M}C_{\bar{S}}(\bar{x}_n) \leq C_{\bar{G}}(\bar{w}_n)$, where $S \in \mathrm{Syl}_5(G)$ and $C_S(x_n) \in \mathrm{Syl}_5(C_G(x_n))$. So $r_8 = 17 \mid \frac{|\bar{M}|}{|C_{\bar{M}}(\bar{x}_n)|} - 1 = 5^a - 1$ and $5^a \leq \frac{|C_{\bar{G}}(\bar{w}_n)|_5}{|C_{\bar{G}}(\bar{x}_n)|_5} = \frac{|cl_{\bar{G}}(\bar{x}_n)|_5}{|cl_{\bar{G}}(\bar{w}_n)|_5} \leq 5^4$. Thus $a = 0$ and hence $\bar{M} \leq C_{\bar{G}}(\bar{x}_n)$. This shows that $|\bar{M}| \leq |PSU_8(2)|_5$, which leads us to get a contradiction by using (9). Therefore, $\bar{M} \cap C_{\bar{G}}(\bar{x}_n) = \{1\}$.

Now let $r \in \pi - \{p\}$ and $K_r \in \mathrm{Syl}_r(K)$. If $y \in K_r \cap C_G(x_n)$, then Lemma 2.14(ii) shows that there exist a divisor $1 \neq m_1$ of n and a divisor β of $\gcd(m_1,q+1)$ such that $|cl_G(y)| = \frac{|GU_n(q)|}{\beta|GL^\epsilon_{n/m_1}(q^{m_1})|}$, where $\epsilon = \mathrm{sgn}((-1)^{m_1})$. Lemma 2.3(i) shows that $\frac{|GU_n(q)|_{p'}}{|\beta|_{p'}|GL^\epsilon_{n/m_1}(q^{m_1})|_{p'}} < |K|_r \leq |G|_r$, which is impossible by considering (8) and the different values of n, m and r. Thus $K_r \cap C_G(x_n) = \{1\}$. On the other hand, Lemma 2.3(ii) guarantees the existence of a u-Sylow subgroup M_u of M such that $M_u \leq N_G(K_r)$ and $x_n \in N_G(M_u K_r)$. Since $\bar{M} \cap C_{\bar{G}}(\bar{x}_n) = \{1\}$, we get that $M_u \cap C_G(x_n) = \{1\}$. Thus $\langle x_n \rangle$ acts fixed-point-freely on $M_u K_r - \{1\}$, so $M_u K_r$ is nilpotent. Therefore, $K_r \leq N_G(M_u)$. Also, the Frattini argument shows that $G = MN_G(M_u) = KM_uN_G(M_u) = KN_G(M_u) = K_pK_rN_G(M_u) = K_pN_G(M_u)$, so $[G : N_G(M_u)]$ is a p-number and hence, for every $1 \neq z \in M_u$,

$$\frac{|cl_G(z))|[C_G(z) : C_{N_G(M_u)}(z)]}{[G : N_G(M_u)]} = |cl_{N_G(M_u)}(z)| < |M_u| = |\bar{M}|_u \leq |cl_{\bar{G}}(\bar{x}_n)|_u.$$

This gives that $|cl_G(z)|_{p'} < |PSU_n(q)|_u$, which is contradiction to Lemma 2.17(iii). This shows that $\bar{M}$ is non-abelian.

By Step 7, $\bar{M}$ is not abelian. Thus $\bar{M} = P_1 \times \ldots \times P_m$, where P_is are non-abelian isomorphic simple groups.

Step 8. $r_{n-1} \in \pi(\bar{M})$. In particular, $\bar{M}$ contains an r_{n-1}-element, say $\bar{x}_{n-1}$. Also, if n is prime, then $r_n \in \pi(\bar{M})$ and $\bar{M}$ contains an r_n-element, say $\bar{x}_n$.

Proof. [6, Step 5] On the contrary suppose that $r_{n-1} \notin \pi(\bar{M})$. Obviously, there exists $1 \leq j \leq m$ such that $P_1^{\bar{x}_{n-1}} = P_j$. Let $j \neq 1$. Thus we can assume that $\{P_1, \cdots, P_{r_{n-1}}\}$ is an $\bar{x}_{n-1}$-orbit. Fix $\bar{g}_i \in P_i$ such that $\bar{g}_1$ is an arbitrary element in P_1 and if $1 \leq i \leq r_{n-1} - 1$, then $\bar{g}_{i+1} = \bar{g}_i^{\bar{x}_{n-1}}$ and otherwise, $\bar{g}_i = K$. Hence $\bar{y} = \prod_{i=1}^{m} \bar{g}_i \in C_{\bar{G}}(\bar{x}_{n-1})$. Thus $C_{\bar{G}}(\bar{x}_{n-1})$ contains a subgroup H isomorphic to P_1, so Lemma 2.20(vii) forces P_1 to be nilpotent, which is a contradiction. Therefore, $j = 1$ and hence, $\bar{x}_{n-1} \in N_{\bar{G}}(P_1)$ and $\bar{x}_{n-1} \notin C_{\bar{G}}(P_1)$. Thus we can assume that $\bar{x}_{n-1} \in \mathrm{Aut}(P_1)$. So $r_{n-1} \mid |\mathrm{Out}(P_1)|$ and $r_{n-1} \nmid |P_1|$. We thus get that P_1 is a non-abelian simple group of Lie type and the r_{n-1}-Sylow subgroups of $\mathrm{Aut}(P_1)$ are isomorphic to $\langle \phi \rangle$, where ϕ is a field automorphism of P_1. Thus Lemma 2.16(i) forces $C_{P_1}(\phi)$ to be nilpotent, which is a contradiction. This shows that $r_{n-1} \in \pi(\bar{M})$ and hence, $\bar{M}$ contains an r_{n-1}-element, say $\bar{x}_{n-1}$.

If n is prime, then the same reasoning as above shows that $r_n \in \pi(\bar{M})$ and $\bar{x}_n \in \bar{M}$. □

Step 9. $\bar{M}$ is a simple group, $C_{\bar{G}}(\bar{M}) = 1$ and $\bar{M} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{M})$.

Proof. [6, Step 6] We first show that $m = 1$. If not, then we can assume that $\bar{x}_{n-1} \in P_2$, so $C_{\bar{G}}(\bar{x}_{n-1})$ contains a subgroup H isomorphic to P_1 and hence, Lemma 2.20(vii) forces P_1 to be nilpotent, which is a contradiction. Therefore, $m = 1$ and hence, $\bar{M}$ is a simple group. Since $\bar{x}_{n-1} \in \bar{M}$, $C_{\bar{G}}(\bar{M}) \leq C_{\bar{G}}(\bar{x}_{n-1})$. Thus Lemma 2.16(i) yields that $C_{\bar{G}}(\bar{M})$

is a normal and nilpotent subgroup of $\bar{G}$. So Step 7 forces $C_{\bar{G}}(\bar{M}) = 1$. We thus get $\bar{M} \trianglelefteq \bar{G} = \frac{N_{\bar{G}}(\bar{M})}{C_{\bar{G}}(\bar{M})} \lesssim \mathrm{Aut}(\bar{M})$, as desired. $\Box$

Step 10. $\bar{M}$ is a simple group of Lie type in characteristic p.

Proof. By Step 9, $\bar{M}$ is a simple group. The classification of finite simple groups shows that one of the following cases occurs:

(i) If $\bar{M}$ is a sporadic simple group, then $|\mathrm{Out}(\bar{M})|$ divides 2 and hence, $\pi(\bar{M}) \cup \pi = \pi(PSU_n(q))$. So $|G|_{r_n} = |\bar{M}|_{r_n}$ and $|G|_{r_{n-1}} = |\bar{M}|_{r_{n-1}}$. Therefore, $\bar{x}_n, \bar{x}_{n-1} \in \bar{M}$. Lemma 2.20(vi) now leads to $|\frac{(q^n-(-1)^n)}{\gcd(n,q+1)(q+1)}|_{\pi'} \mid |C_{\bar{G}}(\bar{x}_n)|$ and either $|\frac{(q^{n-1}-(-1)^{n-1})}{\gcd(n,q+1)}|_{\pi'} \mid |C_{\bar{G}}(\bar{x}_{n-1})|$ or $(q,n) \in \{(3,3),(2,4)\}$, which is impossible by considering the sporadic simple groups.

(ii) If $\bar{M} \cong Alt_u$, the alternating group of degree u, then $|\mathrm{Out}(\bar{M})|$ is a 2-number, so $r_n, r_{n-1} \in \pi(\bar{M})$. First let $(n,q) \neq (4,2),(3,3),(3,4)$. Since $n \geq 3$, $\tau(n) \mid r_n - 1$ and $\tau(n-1) \mid r_{n-1} - 1$, $u \geq 7$. So $\mathrm{Aut}(\bar{M})$ is isomorphic to the symmetric group of degree u, Sym_u. Therefore, $\bar{G} \in \{Alt_u, Sym_u\}$, by Step 9. Without loss of generality, we can assume that $\bar{x}_{n-1} = (1 \cdots r_{n-1})$, a cyclic permutation of length r_{n-1}. Thus if $\bar{G} = Alt_u$, then $C_{\bar{G}}(\bar{x}_{n-1}) = Alt_{u-r_{n-1}} \times \mathbb{Z}_{r_{n-1}}$ and if $\bar{G} = Sym_u$, then $C_{\bar{G}}(\bar{x}_{n-1}) = Sym_{u-r_{n-1}} \times \mathbb{Z}_{r_{n-1}}$. From Lemma 2.20(vii), $C_{\bar{G}}(\bar{x}_{n-1})$ is nilpotent. Thus either $\bar{G} = Alt_u$ and $u - r_{n-1} \leq 3$ or $\bar{G} = Sym_u$ and $u - r_{n-1} \leq 2$, so

$$|C_{\bar{G}}(\bar{x}_{n-1})| \in \{r_{n-1}, 2r_{n-1}, 3r_{n-1}\}. \tag{10}$$

On the other hand, Lemma 2.20(vi) implies that $|\frac{(q^{n-1}-(-1)^{n-1})}{\gcd(n,q+1)}|_{t'}$ divides $|C_{\bar{G}}(\bar{x}_{n-1})|$, where either $t = 1$ or $n \geq 8$ and $(q,t) \in \{(2,3),(3,2),(4,5),(8,3),(7,2)\}$. Since $\pi(\bar{M}) \cup \pi = \pi(PSU_n(q))$ and $(n,q) \neq (3,4)$, we can see that $n-1$ is an odd prime and $\frac{(q^{n-1}+1)}{(q+1)\gcd(n-1,q+1)} = r_{n-1}$. If $\gcd(n-1,q+1) = 1$, then there exists a prime r such that $\frac{(q^{n-1}+1)+(q+1)}{2(q+1)} = (r_{n-1} + 1)/2 < r < r_{n-1} = \frac{(q^{n-1}+1)}{(q+1)}$, by [13, Lemma 1]. On the other hand, $r \in \pi(\bar{M}) \subseteq \pi(PSU_n(q))$ and hence there exists $1 \leq m \leq n$ such that $m \neq n-1$ and $r \in Z_m(-q)$. This forces $(n,q) = (4,3)$ and hence $K = \{1\}$, $r_{n-1} = 7$ and $7 \leq u \leq 10$. But $|PSU_4(3)|$ does not divide $|Alt_u|$ and $|Sym_u|$, which is a contradiction, because $G \cong Alt_u$ or Sym_u. Now let $\gcd(n-1,q+1) \neq 1$. Since $n-1$ is prime, $\gcd(n-1,q+1) = n-1$. Also, $\gcd(n-1,q+1)$ and $\frac{q+1}{\gcd(n,q+1)}$ divide $\frac{(q^{n-1}+1)}{\gcd(n,q+1)}$. Thus (10) shows that $n-1 \in \{1,2,3,5\}$ and hence, we can check that $n = 4$. So $K = \{1\}$, by Step 3 and $s = 1$. Therefore, $G \cong Alt_u$ or Sym_u. Also, (10) forces $\frac{q+1}{\gcd(n,q+1)} \in \{1,2,3\}$. This shows that $q \in \{5,11\}$. If $q = 5$, then $r_{n-1} = 7 \leq u \leq$

$10 = r_{n-1}+3$. But $|PSU_4(5)|$ does not divide $|Alt_u|$ or $|Sym_u|$, where $7 \leq u \leq 10$, which is impossible. Moreover, if $q = 11$, then $r_{n-1} = 37$. Thus $\pi(Alt_{37}) \subseteq \pi(G) = \pi(PSU_4(11))$, which is a contradiction. Now let $(q, n) = (2, 4)$ and $|\frac{(q^n-(-1)^n)}{(q+1)\gcd(n,q+1)}| = 5$. Step 3 shows that $K = \{1\}$ and by the above statements, $r_{n-1} = 5 \leq u \leq 8 = r_{n-1} + 3$. On the other hand, $|PSU_4(2)| = |G| = |Alt_u|$ or $|Sym_u|$, by Remark 2.18, which is a contradiction. The same reasoning rules out the case $(n, q) = (3, 3)$ and $(3, 4)$.

(iii) Let $\bar{M}$ be a simple group of Lie type in characteristic t, where $t \in \pi(\bar{M})$. On the contrary, suppose that $t \neq p$. By [18], there exists $u \in \pi(\bar{M}) - \{t\}$ such that $\bar{M}$ does not contain any element of order tu and hence, there exists a u-element $\bar{w} \in \bar{M}$ such that $|cl_{\bar{M}}(\bar{w})|_t = |\bar{M}|_t$. But $|cl_{\bar{M}}(\bar{w})|$ divides $|cl_M(w)|$ and $|cl_M(w)|$ divides $|cl_G(w)|$. Thus $|\bar{M}|_t$ divides $|PSU_n(q)|_t$. Since $\bar{x}_{n-1} \in \bar{M} \trianglelefteq \bar{G}$, $|cl_{\bar{G}}(\bar{x}_{n-1})| < |\bar{M}|$. Considering the order of finite simple groups of Lie type in characteristic t shows that $|\bar{M}| \leq (|\bar{M}|_t)^3$. Since $KC_G(x_{n-1}) \leq G$, we deduce that $|K/C_K(x_{n-1})|_p$ divides $|cl_G(x_{n-1})|_p = p^{n(n-1)k/2}$. On the other hand, Lemma 2.4(v) gives that if $|K/C_K(x_{n-1})|_p = p^\gamma$, then $\tau(n-1)k \mid \gamma$. Thus if $\tau(n-1) = 2(n-1)$ and $\tau(n) = n/2$, then $q^{n-1} \mid [G : KC_G(x_{n-1})] = |cl_{\bar{G}}(\bar{x}_{n-1})|$ and if $\tau(n-1) = (n-1)$ and $\tau(n) = 2n$, then $q^{(n-1)/2} \mid [G : KC_G(x_{n-1})] = |cl_{\bar{G}}(\bar{x}_{n-1})|$. Therefore,

$$q^i|cl_G(x_{n-1})|_{\pi'} \leq |\bar{M}| < (|\bar{M}|_t)^3 \leq (|PSU_n(q)|_t)^3, \tag{11}$$

where if $\tau(n-1) = 2(n-1)$, $\tau(n) = n/2$ and $p \in \pi$, $i = n-1$, if $\tau(n-1) = (n-1)$, $\tau(n) = 2n$ and $p \in \pi$, $i = (n-1)/2$ and otherwise, $i = 0$. Thus considering (11), the conditions obtained in Steps 3, 6, Lemma 2.16(i) and the order of finite simple groups of Lie type in characteristic t force

A. $O_\pi(G) = O_p(G)$ and $(n, q, t) \in \{(10, 2, 3), (9, 3, 2), (j, 2, 3), (j, 3, 2), (6, 4, 5) : j \in \{6, 8\}\} - \{(8, 3, 2), (6, 2, 3)\}$

or

B. $O_\pi(G) = 1$ and

$$(n, q, t) \in \{(5, 2, 3), (5, 3, 2), (4, 8, 3), (4, 7, 2), (4, 2, 3), (4, 3, 2), (3, 3, 2), (3, 4, 5), (3, 7, 2), (3, 8, 3)\}$$

or

C. $ps \mid |O_\pi(G)| = |K|$ and

$$(n, q, s, t) = (8, 2, 3, 43).$$

If $O_\pi(G) = O_p(G)$ and $(n, q, t) = (8, 2, 3)$, then $43 = r_{n-1} \in \pi(\bar{M}) \subseteq \pi(PSU_n(q)) \subseteq \{2, 3, 5, 7, 11, 17, 43\}$, which is impossible by considering [20, Table 1]. The same reasoning rules out the case when $O_\pi(G) = O_p(G)$ and $(n, q, t) \in \{(10, 2, 3), (6, 4, 5), (6, 3, 2)\}$ or $O_\pi(G) = \{1\}$ and $(n, q, t) \in \{(5, 2, 3), (5, 3, 2), (4, 8, 3), (4, 7, 2), (4, 2, 3), (3, 4, 5), (3, 7, 2), (3, 8, 3)\}$. If $O_\pi(G) = O_p(G)$ and $(n, q, t) = (9, 3, 2)$, then since $|\bar{M}|_2 \leq |G|_2 \leq (|PSU_9(3)|_2)^2 = 2^{46}$ and $\bar{M} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{M})$, we can see that $547 = r_7(-3) \in \pi(\bar{M})$. Therefore, there exists $1 \leq m \leq 2.46$ such that $547 \in Z_m(2)$, which is a contradiction, because $\exp_{547}(2) > 2.46$. If $O_\pi(G) = \{1\}$ and $(n, q, t) = (4, 3, 2)$, then Remark 2.18 and, Steps 8 and 9 show that $|G| = |PSU_4(3)| = 2^7.3^6.5.7$, $7 = r_3 \in \pi(\bar{M})$ and $\bar{M} \trianglelefteq \bar{G} = G \lesssim \mathrm{Aut}(\bar{M})$, which is impossible by considering [20, Table 1].

Now let $ps \mid |O_\pi(G)| = |K|$ and $(n, q, s, t) = (8, 2, 3, 43)$. Then $43 = r_7 \in \pi(\bar{M}) \subseteq \pi(G) = \pi(PSU_n(q)) \subseteq \{2, 3, 5, 7, 11, 17, 43\}$ and by Lemma 2.14, $|G|_{43} = |PSU_n(q)|_{43}$. Therefore, [20, Table 1] forces $\bar{M} \cong PSL_2(43)$, so $5 \in \pi(K) \cup \pi(\mathrm{Out}(\bar{M})) = \pi(O_{\{2,3\}}(G)) \cup \pi(\mathbb{Z}_2)$, which is a contradiction.

If $O_\pi(G) = \{1\}$ and $(n, q, t) = (3, 3, 2)$, then $7 = r_3 \in \pi(\bar{M}) \subseteq \pi(G) = \pi(PSU_n(q)) = \{2, 3, 7\}$, so [20, Table 1] shows that $\bar{M} \cong PSL_3(2)$ or $PSL_2(8)$ and hence, since $\bar{M} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{M})$, $2, 3 \in \pi(\mathrm{Out}(\bar{M})) = \pi(\mathbb{Z}_2)$ or $\pi(\mathbb{Z}_3)$, which is impossible.

This shows that $\bar{M}$ is a finite simple group of Lie type in characteristic p, as wanted. □

Step 11. $\bar{M}$ is isomorphic to $PSU_n(q)$.

Proof. For a finite group H, fix $\varphi(H) = \max\{\exp_u(p) : u \in \pi(H) - \{p\}\}$ and $\psi(H) = \max\{\exp_u(p) : u \in \pi(H) - (Z_{\varphi(H)}(p) \cup \{p\})\}$.

We claim that $r_n \in \pi(\bar{M})$. On the contradiction, suppose that $r_n \notin \pi(\bar{M})$. Since $r_n \nmid |K|$, $\bar{M} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{M})$ and $\bar{M}$ is a simple group of Lie type over a field with p^e elements, by Steps 9 and 10, we conclude that $r_n \mid e$. If n is odd, then $\varphi(G) = \tau(n)k$ and since $\tau(n)k = 2nk \mid r_n - 1$, we get from considering the order of finite simple groups of Lie type over a field with p^e elements that $\pi(\bar{M})$ contains a prime divisor u such that $\exp_u(p) \geq e \geq r_n > \tau(n)k = \varphi(G)$, which is a contradiction. Now let n be even. Since by Step 8, $r_{n-1} \in \pi(\bar{M})$, we have $\varphi(G) = \varphi(\bar{M}) = \tau(n-1)k$ and hence, considering the order of finite simple groups of Lie type over a field with p^e elements shows that $e \mid \tau(n-1)k = 2(n-1)k$. Thus $r_n \mid (n-1)k$. On the other hand, $\tau(n)k \mid r_n - 1$ and $r_n - 1$ is even, so $nk \mid r_n - 1$. This yields $nk < (n-1)k$, a contradiction. Therefore, $r_n \in \pi(\bar{M})$, as wanted. Thus

$$\varphi(G) = \varphi(\bar{M}) = \begin{cases} \tau(n)k, & \text{if either } n \text{ is odd or } (n,q) = (4,2) \\ \tau(n-1)k, & \text{otherwise} \end{cases}. \tag{12}$$

If $(n,q) = (4,2)$, let $r = 3$, if $(n,q) \in \{(5,2),(6,2)\}$, let $r = 5$, if $n > 6$ is even, let $r \in Z_{2(n-3)k}(p)$, if $n \leq 6$ is even and $(n,q) \neq (4,2),(6,2)$, let $r \in Z_{nk}(p)$ and if n is odd and $(n,q) \neq (5,2)$, let $r \in Z_{2(n-2)k}(p)$. If $r = 2$, then obviously $r \in \pi(\bar{M})$. Now let r be odd. By Tables 1 and 2, there exists a natural number m such that $\varphi(\bar{M}) = me$ and hence, if $(n,q) \neq (4,2),(3,3),(5,2),(6,2)$, then we can conclude from (12) that $r \nmid e$, so repeating the above argument shows that $r \in \pi(\bar{M})$. Also, if $(n,q) = (4,2)$, then $|PSU_4(2)| \mid |G|$ and $\pi(G) = \pi(PSU_4(2))$, so since by Steps 3 and 9, $K = \{1\}$ and $M \trianglelefteq G \lesssim \mathrm{Aut}(M)$, we get from [14] that $\bar{M} = M \cong PSU_4(2)$, as wanted in this case. The same reasoning shows that if $(n,q) = (3,3)$, then $\bar{M} = M \cong PSU_3(3)$. If $(n,q) = (5,2)$, then by Step 8, $r = r_4 = r_{\tau(n-1)k} \in \pi(\bar{M})$, as wanted. Finally if $(n,q) = (6,2)$, then since $n-1$ is prime and $q+1 \mid n$, we get that $K = \{1\}$ and hence, $\bar{M} \trianglelefteq \bar{G} = G \lesssim \mathrm{Aut}(\bar{M})$. But $\pi(G) = \pi(PSU_6(2))$ and $|PSU_6(2)| \mid |G|$, so [20, Table 1] forces $\bar{M} \cong PSU_6(2)$, as wanted. Thus we can assume that $(n,q) \neq (4,2),(3,3),(6,2)$ and $r \in \pi(\bar{M})$. Therefore, since $n \geq 3$, we see that

$$\psi(G) = \psi(\bar{M}) = \begin{cases} \tau(n-2)k, & \text{if } n \text{ is odd and } (n,q) \neq (5,2),(3,3) \\ 4, & \text{if } (n,q) = (5,2) \\ nk, & \text{if } n \leq 6 \text{ is even and } (n,q) \neq (4,2),(6,2) \\ \tau(n-3)k, & \text{if } n > 6 \text{ is even} \end{cases}.$$

Since $\bar{M}$ is isomorphic to one of the simple groups mentioned in Tables 1 and 2, comparing the above values for $\varphi(\bar{M})$ and $\psi(\bar{M})$ and the values obtained in Tables 1 and 2, and considering the fact that $\pi(\bar{M}) \subseteq \pi(G) = \pi(PSU_n(q))$ show that $\bar{M} \cong PSU_n(q)$, as desired. □

Step 12. $K = \{1\}$.

Proof. Since $\bar{x}_n \in \bar{M}$, $|cl_{\bar{M}}(\bar{x}_n)|$ divides $|cl_{\bar{G}}(\bar{x}_n)|$. On the other hand, $|cl_{\bar{G}}(\bar{x}_n)|$ divides $|cl_G(x_n)|$ and $|cl_{\bar{M}}(\bar{x}_n)| = \frac{|GU_n(q)|}{(q^n-(-1)^n)}$. Thus since $\frac{|GU_n(q)|}{(q^n-(-1)^n)}$ is maximal in $cs(G)$ by divisibility, by Lemma 2.13(i), we get that $|cl_G(x_n)| = |cl_{\bar{G}}(\bar{x}_n)|$ and hence, Lemma 2.4(iv) forces $\frac{|G|}{|C_G(x_n)|} = \frac{|G|}{|KC_G(x_n)|}$. Therefore, $C_G(x_n)K = C_G(x_n)$, so $K \leq C_G(x_n)$. Thus $N \leq C_G(x_n)$.

H	${}^2D_m(p^e)$, $D_{m+1}(p^e)$ ($m \geq 4$) $B_m(p^e)$, $C_m(p^e)$ ($m \geq 2$)	$A_{m-1}(p^e)$	${}^2A_{m-1}(p^e)$, (m is odd)
$\varphi(H)$	4, if $(m, p^e) = (3, 2)$ $2me$, otherwise	5, if $(m, p^e) = (6, 2)$ 1, if $(m, p^e) = (2, 2^u - 1)$ me, otherwise	2, if $(m, p^e) = (3, 2)$ $2me$
$\psi(H)$	3, if $(m, p^e) = (3, 2)$ 4, if $(m, p^e) = (4, 2)$ 1, if $(m, p^e) = (2, 2^u - 1)$ $2(m-1)e$, otherwise	4, if $(m, p^e) = (6, 2)$ 5, if $(m, p^e) = (7, 2)$ $-$, if $(m, p^e) = (2, 2^u - 1)$ $(m-1)e$, otherwise	$-$, if $(m, p^e) = (3, 2)$ 1, if $(m, p^e) = (3, 2^u - 1)$ 4, if $(m, p^e) = (5, 2)$ $2(m-2)e$, otherwise
H	$E_6(p^e)$	$E_7(p^e)$	$E_8(p^e)$
$\varphi(H)$	$12e$	$18e$	$30e$
$\psi(H)$	$9e$	$14e$	$24e$

Table 1: $\phi(H)$ and $\psi(H)$, where H is a finite simple group of Lie type over a field with p^e elements

H	${}^2A_{m-1}(p^e)$, (m is even)	$F_4(p^e)$	$G_2(p^e)$	${}^2E_6(p^e)$	${}^3D_4(q^3)$	${}^2B_2(2^e)$	${}^2F_4(2^e)$	${}^2G_2(3^e)$
$\varphi(H)$	4, if $(m, p^e) = (4, 2)$ 1, if $(m, p^e) = (2, 2^u - 1)$ $2(m-1)e$, otherwise	$12e$	$6e$	$18e$	$12e$	$4e$	$12e$	$6e$
$\psi(H)$	4, if $(m, p^e) = (6, 2)$ 2, if $(m, p^e) = (4, 2)$ $-$, if $(m, p^e) = (2, 2^u - 1)$ me, if $m \leq 6$, $(m, p^e) \neq (2, 2^u - 1), (6, 2), (4, 2)$ $2(m-3)e$, otherwise	$8e$	$3e$	$12e$	$6e$, if $p^e \neq 2$ 3, otherwise	e	$6e$	e

Table 2: $\phi(H)$ and $\psi(H)$, where H is a finite simple group of Lie type over a field with p^e elements

On the other hand, obviously, $N \leq O_s(G)$. Thus for $S \in \mathrm{Syl}_s(G)$, $1 \neq Z(S) \cap N \leq C_G(x_n)$, which is a contradiction with Lemma 2.20(i), because Step 3 shows that either $s = p$ or $\{q, s\} = \{2, 3\}$. Therefore, $K = \{1\}$, as desired. □

Step 13. $G = M \cong PSU_n(q)$.

Proof. Since by Steps 9, 11 and 12, $K = \{1\}$, $M \trianglelefteq G \lesssim \mathrm{Aut}(M)$ and $M \cong PSU_n(q)$, Theorem 2.25 shows that $G = M \cong PSU_n(q)$, as desired. □

The proof of the main theorem is complete.

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References

[1] N. Ahanjideh, Finite groups with the same conjugacy class sizes as a finite simple group, *Int. J. Group Theory* (In Press).

[2] N. Ahanjideh, On Thompson's conjecture for some finite simple groups, *J. Algebra* **344** (2011) 205-228.

[3] N. Ahanjideh, Thompson's conjecture for some finite simple groups with connected prime graph, *Algebra Logic* **51 (6)** (2013) 451-478.

[4] N. Ahanjideh and M. Ahanjideh, On the validity of Thompson's conjecture for finite simple groups, *Commun. Algebra* **41 (11)** (2013) 4116-4145.

[5] N. Ahanjideh, On the Thompson's conjecture on conjugacy classes sizes, *Int. J. Algebr Comput.* **23 (1)** (2013) 37-68.

[6] N. Ahanjideh, Thompson's conjecture for finite simple groups of Lie type B_n and C_n, *J. Group Theory* **19 (4)** (2016) 713-733.

[7] R.W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley, New York, 1985.

[8] G. Chen, On Thompson's conjecture, *J. Algebra* **185** (1996) 184-193.

[9] I.B. Gorshkov, Thompson's conjecture for simple groups with connected prime graph, *Algebra Logic* **51 (2)** (2012) 111-127.

[10] E.I. Khukhro and V.D. Mazurov, *Unsolved problems in group theory: The Kourovka Notebook*, Sobolev Institute of Mathematics, Novosibirsk, 17th edition, 2010.

[11] E.I. Khukhro, *Nilpotent groups and their automorphisms*, De Gruyter, Berlin, 1993.

[12] P. Kleidman and M. Liebeck, *The subgroup structure of finite classical groups*, Cambridge University Press, 1990.

[13] A.S. Kondratev and V.D. Mazurov, Recognition of alternating groups of prime degree from their element orders, *Siberian Math. J.* **41 (2)** (2000) 294-302.

[14] M. Herzog, On finite simple groups of order divisible by three primes only, *J. Algebra* **10** (1968) 383-388.

[15] A.V. Vasilev, On Thompson's conjecture, *Sib. El. Mat. Izv.* **6** (2009) 457-464.

[16] A.V.Vasilev, On finite groups isospectral to simple classical groups, *J. Algebra* **423** (2015) 318-374.

[17] A.V. Vasilev and M.A. Grechkoseeva, On recognition by spectrum of finite simple linear groups over fields of characteristic 2, *Siberian Math. J.* **46 (4)** (2005) 593-600.

[18] A.V. Vasilev and E.P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic* **44 (6)** (2005) 381-406.

[19] A.V. Zavarnitsine, Recognition of the simple groups $L_3(q)$ by element orders, *J. Group Theory* **7** (2004) 81-97.

[20] A.V. Zavarnitsine, Finite simple groups with narrow prime spectrum, *Siberian Electronic Mathematical Reports* **6** (2009) 1-12.

[21] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math.Phys.* **3** (1892) 265-284.